NEW POSSIBILITIES FOR REPRESENTATIONAL ACTIVITY IN STEM EDUCATION WITH 3D-PRINTED MANIPULATIVES

Teo Paoletti, PhD
University of Delaware
teop@udel.edu

Steven Greenstein, PhD
Montclair State University
greensteins@montclair.edu

Madhavi Vishnubhotla
Montclair State University
vishnubhotm1@montclair.edu

Mustafa Mohamed
Montclair State University
mohamedm3@montclair.edu

Abstract

Although educational manipulatives have a long history, increased access to 3D printing has opened new possibilities for their design and use in teaching and learning. In this paper, we describe how 3D design and printing technologies provided innovative ways of supporting middle-school students in constructing and reasoning about covarying quantities, which is central to students' abilities to make sense of situations and model dynamic phenomena (e.g., Thompson 2008).

The tasks described in this paper are part of a larger effort consisting of small-group and whole-class teaching experiments (Cobb et al., 2003; Steffe & Thompson, 2000) with the objective of designing tasks that support middle-school students' developing meanings for graphs and various function classes via their quantitative and covariational reasoning. In this report we describe two tasks, the *Cone Task* and the *Growing Triangle Task*, for which 3D printing proved useful in our efforts to support students' reasoning. We first detail several iterations in the development of the *Cone Task* to highlight issues we perceived in students' activity including how and why we conjectured 3D printing may help us alleviate certain issues. We then describe our design of the *Growing Triangle Task* which explicitly included the use of 3D manipulatives based on our prior success in the *Cone Task*.

Prior to describing the tasks and manipulatives, we outline the theories we coordinated to frame the study: quantitative reasoning, covariational reasoning, and designing for mathematical abstraction. We then describe each task in detail and include aspects of students' engagement with those tasks via manipulatives that mediated (Vygotsky, 1978) their sense making, particularly their constructions of quantities and relationships between them. We then discuss connections we made between the designs of our tasks and tools and the theories that informed them. We conclude with implications for learning and teaching in STEM fields.

1. Theoretical Perspectives

1.1. Quantitative and Covariational Reasoning

Steffe, Thompson, and colleagues' (Steffe, 1991; Smith III & Thompson, 2008; Thompson 2008) stance on quantitative reasoning underscores that teachers and researchers cannot assume students maintain understanding of quantities in ways compatible with their intentions, because quantities are constructed by each of us in order to make sense of our experiential world (Glasersfeld, 1995). Steffe (1991) explains:

> Properties of concepts are introduced by the knowing subject's actions in the construction of the concepts. For example, the sweetness of sugar is introduced by the tasting individual just as the color of roses is introduced by the viewing individual. The idea that an activity introduced by the child in constructing a concept might be later abstracted as a property of the concept that could be called "quantity" is an important one. It changes the emphasis from viewing the sources of quantity as existing in what we take to be external to the child to taking quantity as emerging from the child's interactions with elements in his or her environment. (p. 62)

As conceptual entities, understandings of a quantity may differ from individual to individual. Therefore, it is critical to attend to *students'* conceptions of quantities, since the mathematical representations they construct may be consistent with their own conceptions but also inconsistent with the target quantities that teachers and researchers intend for them to explore (Author, 2015; Moore & Carlson, 2012).

There are two aspects of quantitative reasoning we pay particular attention to in this paper. First, quantitative reasoning can involve numerical and non-numerical reasoning (Johnson, 2012), but the essence of quantitative reasoning is non-numerical (Smith III & Thompson, 2008), that is, having more to do with developing meanings for a situation than introducing or operating on numeric values. Second, Castillo-Garsow et al. (2013) differentiate between students' conceiving quantities varying smoothly (e.g., imagining the height of liquid in a bottle changing smoothly, as if being filled by a spigot) versus in chunks (e.g., imagining the height of liquid in a bottle changing in chunks, as if being filled cup by cup). These researchers note that reasoning about smoothly (or continuously) changing quantities is non-trivial. Thus, it is important to provide students repeated opportunities to construct and reason about smoothly changing phenomena.

Building on prior characterizations of quantitative reasoning, Carlson et al. (2002) described covariational reasoning as entailing a student coordinating two quantities with attention to how the quantities change together. They specified mental actions that allow for a fine-grained analysis of students' activity, including coordinating *direction of change* (area increases as base length increases; MA2) and *amounts of change* (the *change* in area *increases* as base length *increases in equal successive amounts*; MA3). Researchers (Authors, 2018; Moore, 2014) have shown that students can leverage these mental actions to construct and represent relationships between covarying quantities in productive ways.

1.2. Designing for Mathematical Abstraction

Due to the aforementioned importance of attending to students' conception of quantities, we engaged in a process of designing for mathematical abstraction (Pratt & Noss, 2010). Our goal was to design physical manipulatives with particular affordances for the kinds of embodied (i.e., body-based) interactions (Varela et al., 1992) that could enable students to construct the quantities we aimed to investigate. Relying on a Piagetian (1970a) perspective of cognitive development, Author (2018) noted, "The faithful mental representation of objects is critical, because conceptual thought proceeds from representational thought and representational thought proceeds from perception" (p. 3). Hence, by making quantities available to students for abstraction through their sensorimotor engagement (Kamii & Housman, 2000; Piaget, 1970b) with both digital and physical representations, we hoped to provide them with opportunities to coordinate meanings across these representations in a way that would support them in constructing the target quantities.

2. Theory-Driven Task Design

We now describe *The Cone Task* and *The Growing Triangle Task*, which we iteratively tested and refined. For each of the tasks, we describe the different representations we made available to students. We leverage the elements of our theoretical framework along with excerpts of student engagement to describe the affordances and limitations of the representations. We pay particular attention to those affordances we aimed to embed in the design of new manipulatives through an approach we refer to as *responsive design*. Responsive design refers to the process by which access to 3D design and printing enabled us to create manipulatives in response to what we learned about students' thinking in ways that resonate with how a teacher might engage in professional noticing (Sherin, et al., 2011) as they anticipate, attend to, and interpret classroom interactions.

Finally, we use *representational activity*[1] to refer to students' mediated engagement with one or more representations and the ways in which learners perceive, interpret, and coordinate attributes across representations. As an example, consider a student presented with a physical (right) cone and a picture of a (two-dimensional) cross-section of the cone. The student is tasked with determining a method for finding its slant height. Upon perceiving an attribute of the physical cone and interpreting it as the cone's slant height, the student then refers to the picture to find slant height in that representation. The student perceives an attribute of the cone's triangular cross-section and interprets it as the same slant height that was identified in the physical representation. From there, the student determines that they can use the Pythagorean theorem to determine the slant height of the physical cone. Thus, the student's success in developing a formula was made possible by coordinating interpretations across the two representations.

2.1. The Cone Task

The *Cone Task* (hyperlink blinded; Figure 1) was adapted from Authors' (2017) use in clinical interviews to elicit college students' covariational reasoning. We envisioned that by prompting middle-grades students to describe the relationship between surface area and height of the smoothly growing cone, we could support them in differentiating between increasing, decreasing, or constantly changing amounts of one quantity (surface area) in relation to changes in a second quantity (height). In particular, with Carlson et al.'s (2002) mental actions in mind, we chose this task to support students first in constructing smoothly changing quantities (height and surface area), then in identifying and coordinating amounts of change (hereafter 'AoC') of one quantity (surface area) with respect to equal changes in the other (height). In what follows, we describe several iterations of the task and highlight (1) what we learned about the fidelity of students' representational activity through their mediated engagement with digital and paper-and-pencil representations, (2) our rationales for the design and introduction of 3D manipulatives in light of what we learned, and (3) the ways in which the manipulatives mediated students' constructions of quantities and how their reasoning about those quantities informed further adaptations of the task.

[1] This notion is informed by Maffia and Maracci's (2019) concept of *semiotic interference*.

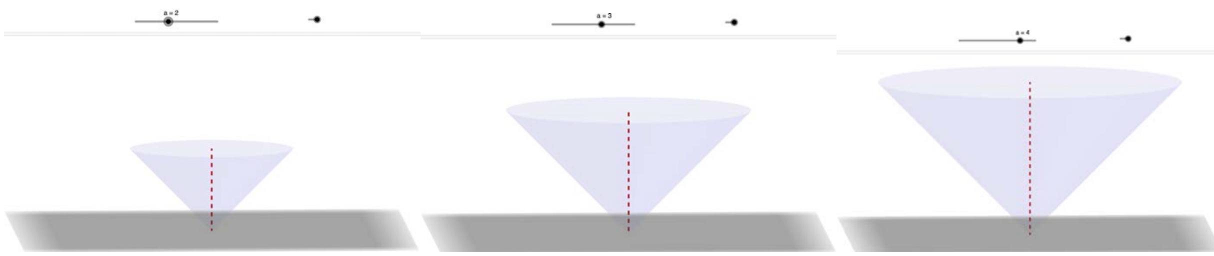


Figure 1: Screenshots of the *Cone Task* applet.

2.1.1. Iteration 1: Applet and handout. For the first iteration of the *Cone Task*, which we gave to two pairs of students, we presented students a GeoGebra applet that shows an infinitesimally thin and growing cone in 3D (Figure 1). The long slider value corresponds to the height of the cone. We also had a short slider that can be used to change how the height of the cone varies: either smoothly or in (chunky) integer increments. In every iteration of this task, as well as in the *Growing Triangle Task* described below, the dynamic animation proved productive in supporting students in imagining and describing smoothly changing quantities. For instance, one student described the situation by explaining that the cone grows "constantly, like there's never a time where it stops growing, it keeps on growing." We conjecture that dynamic representations of continuously changing phenomena in tasks supported students in constructing smooth images of change (Castillo-Garsow et al., 2013). In order to maintain the focus of this article, we do not describe aspects of the students' activity that offer insights into their smooth imagery for each iteration and task.

In addition to the GeoGebra applet, we provided students with a handout of screenshots of the cone in the applet at integer heights (Figure 2). As they engaged with this paper-and-pencil representation, we identified several features of the students' conceptions of the situation that we found to be revealing of their constructions of quantities other than the target ones. As one example, although students' observations indicated they interpreted the situation in 3D (e.g., by describing one quantity as the "circumference of the top of the cone"), they tended to describe or reason about the cone as a 2D object, with several students referring to the cone as a "triangle." As a second example, one pair of students consistently drew what we infer to be changes in the slant height of the cone when explaining that the amount of change of "surface area" was constant. In this latter case, we inferred that rather than constructing mental representations of a 3D cone and then operating on this representation to construct quantities compatible with the cone's height and surface area, these students constructed mental representations of triangles and operated on those representations to construct surface area as a quantity that is equivalent to the slant height of the cone. In other words, the students were using a 1D quantity (slant height) of a 2D representation (a piece of paper) as a surrogate for a 2D quantity (surface area) of a 3D object (a cone).

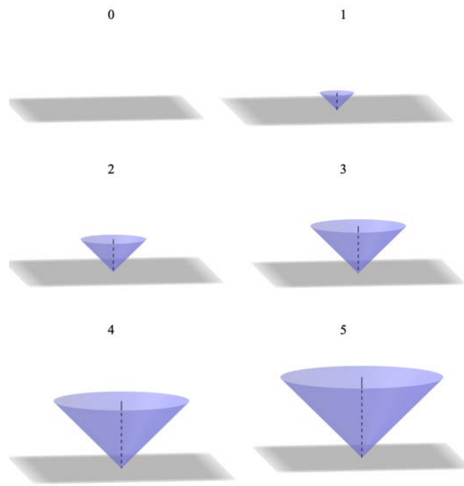


Figure 2: The *Cone Task* handout

When it became apparent the students were not reasoning about surface area as intended, the teacher-researcher (TR), in hoping to support students in constructing surface area of the cone as a 2D quantity related to a 3D shape, asked them to shade and color-coordinate surface areas and changes in surface area on their handouts. We note that such effortful activity (taking between 7 and 9 minutes to complete) was productive in supporting students constructing normatively correct descriptions of how the cone's surface area and height covaried. That said, from the researchers' (and perhaps the students' own) perspective, the ambiguity in the quantity with which the students were operating persisted. For instance, one student described her shaded regions on the 2D handout as "triangles" while simultaneously referring to the "space" the triangles were filling as volume at one point and surface area at another.

In reflecting on why students might have constructed and reasoned about these quantities rather than the target ones, we noted that the students' engagement with a 3D situation was constrained in activities that were mediated by representations on a computer screen and on a pencil-and-paper handout, both of which are 2D in nature. The representation of a 3D cone in the GeoGebra applet can easily be perceived as a triangle, as the 2D representation of a cone is geometrically similar to its triangular (2D) cross-sections. Thus, it is straightforward to imagine how one's perceptual activity might lead them to construct – and subsequently operate on – a mental representation of the cone as a triangle instead.

2.1.2. Iteration 2: Applet, handout, and 3D manipulatives. Due to the aforementioned ambiguities, we were unsure the extent to which students were reasoning about the target quantities. In response, we sought to offer coherence between a 3D phenomenon and a 3D representation of it. Hence, we designed and 3D printed physical manipulatives (Figure 3) to support students' constructions of the two relevant quantities and to examine their reasoning about them. Specifically, we 3D printed (1) entire purple cones intended to represent the growing cone at integer height values (Figure 3, left), and (2) red truncations of the cone intended to represent the chunks of cone added on to the previous cone to obtain the next one (Figure 3, center and right).

Theory informed the design of the manipulatives in the sense that we intended them to have particular affordances for likely manipulations[2] that we hypothesized would enable students to construct the target quantities through their goal-oriented, sensorimotor engagement (Laborde & Laborde, 2011). Specifically, we conjectured the manipulatives would support students in constructing the cone as a 3D object and its surface area as a 2D quantity of this object as they identify and coordinate various perceptual attributes. Once students had constructed such images, we conjectured the truncated red pieces, each with an equal change in height, might support them in identifying increasing AoC in surface area for equal changes in height (MA3).

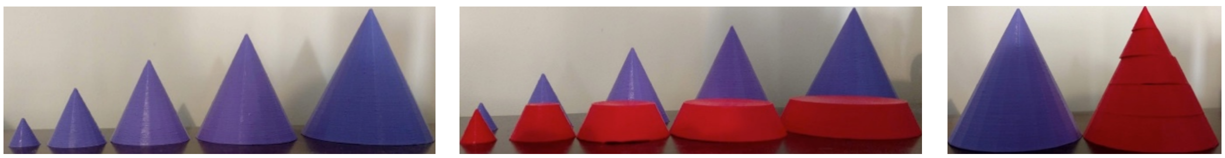


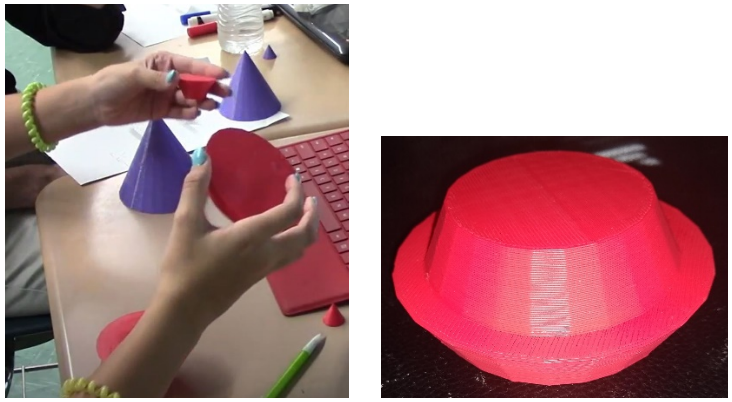
Figure 3: The 3D-printed cone manipulatives

In our second iteration of the *Cone Task*, we made all three representations (the 3D manipulatives, the GeoGebra applet, and the handout) available to two new groups of students (one triple, one single). We again tasked students to describe how the surface area and height of the cone covaried. We identified several features of their representational activity that are notable in relation to the purpose of this paper. First, when presented with the manipulatives yet with no prompting about how to use them, all four students picked up and played with the various manipulatives. Second, each of the students identified the 2D surface area of the 3D cone by enacting pointing and motioning gestures over parts of the manipulatives as they described the cone's outer area as something they could measure. Third, given their demonstrated interest in the 3D manipulatives and the fact that they rarely referred to the 2D handout, we conjecture that the 3D representation of the 3D phenomena not only provided students with an occasion for playful activity, the manipulatives also provoked their selective attention to perceptions of the cone that they deemed essential to surface area as a 2D quantity of a 3D object, as we intended.

Despite these findings, when determining how the surface area and height covaried, we were yet again confronted with the realization that the quantity(ies) students were reasoning about was still often either ambiguous from our perspective or clearly *not* the surface area. As one example, Greta (a student working alone) used the truncated red pieces to accurately describe that the AoC were increasing. Greta used the truncated cones to justify her conclusion, explaining that the truncated cone in her right hand is larger than the one in her left as shown in Figure 4a. However, the quantity Greta was using to make the comparison in this and in other instances was not the cone's outer surface area, our target quantity. Instead, she consistently compared the relative size of the top faces of the red truncations in order to explain that the AoC of surface area increased for equal changes in height (e.g., by comparing the areas of the larger bases of two truncated cones as seen in Figure 4b). Although Greta came to a correct description of the relationship between surface area and height, she was using area of the larger circular face of the truncations to reason about surface area changes. Hence, as in the previous iteration, the student was able to make normatively correct claims regarding how surface area and height

[2] We use "likely manipulations" in the same sense as Verillon and Rabardel's (1995) *utilization schemes*.

covaried despite her not necessarily reasoning about surface area in ways compatible with our intentions.



(a) (b)

Figure 4: (a) Greta compares the sizes of two truncated cones, and (b) one of the stacks Greta used to compare amounts of change in "surface area."

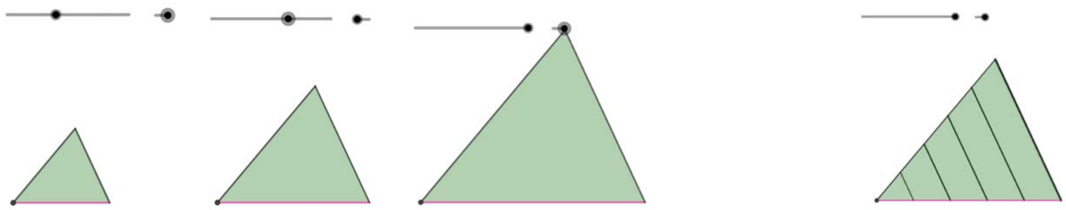
In contrast to Greta, the group of three students tended to rely on intuitive or explicit notions of volume when describing the increasing AoC for equal changes in height. For instance, they would describe the consecutive red pieces as getting "bigger" or "wider" as they gestured with their hands to indicate the entire piece and not just its outer surface. Reflecting across these students' activity, we noted they each readily identified volume as a quantity they could measure in the situation. In considering the affordances of the manipulatives to explain these actions, we conjecture the cone's solid quality (in contrast to the infinitesimally thin cone depicted in the GeoGebra animation) likely evoked this interpretation. Although surface area was "present" in the manipulatives, their volume seemed to be apparent to the students as a more salient feature than their outer surface area. As volume seemed to be a more salient quantity to these students, we wondered whether the same might be true for other students.

2.1.3. Iteration 3: Switching to volume and height. Since our primary goal for this task is to support students in identifying AoC of one quantity for equal changes in the other – not in constructing surface area *per se* – we left the manipulatives and applet unchanged and instead adapted the task itself by asking students to describe how the cone's *volume* varied for equal changes in height. Two groups of students (one pair and one single) engaged with this iteration of the task. Each student reasoned with volume or a proxy quantity (e.g., weight of the manipulative or amount of plastic comprising it). As an example, Ariana, in describing volume in terms of weight, determined that if the red truncated cones have "different sizes, their weights have to be different," and so the "weight is increasing" for consecutive changes in height represented by the red manipulatives. Ariana used this reasoning to claim that as the height of the cone increases by equal amounts, the volume of the cone increases by greater amounts (MA3). In doing so, Ariana's reasoning is consistent with having constructed volume of the cone, AoC of volume of the cone, and height of the cone via her mediated engagement with the manipulatives and applet.

2.2. The Growing Triangle Task

The *Growing Triangle Task* had many purposes. First, and similar to the *Cone Task*, one purpose is to support students in reasoning covariationally with smooth images of change.

Hence, we first have students interact with a dynamic GeoGebra applet showing a smoothly growing scalene triangle (hyperlink blinded; Figure 5a) that can direct their attention to the triangle's smoothly growing area and base length; the triangle's base length increases as the long slider's point continuously moves to the right. With the intention of supporting students in identifying increasing AoC of the triangle's area, we included a second smaller slider that enables the user to increase the increment by which the pink length increases (e.g., to equal integer chunks instead of apparently smoothly). We have the 'trace' option available so that students can visually identify the increasing AoC of area in the applet (i.e., the increasing size of the consecutive trapezoids shown in Figure 5b).



(a) (b)

Figure 5: (a) Three screenshots of the applet's growing triangle and (b) the applet with 'trace' used to highlight AoC of area.

In this task, we intentionally leveraged 3D printing at the outset of our enactment of the *Growing Triangle Task* to use what we had learned about the promise of 3D manipulatives from the *Cone Task*. Our purpose in doing so was to accomplish two goals: (1) supporting students in constructing AoC as a quantity unto itself that could be compared, and (2) supporting students in comparing changes in AoC (hereafter 'AoC of AoC'). These goals were particularly important as our objectives included supporting students in developing productive meanings for quadratic change (Ellis, 2011; Lobato et al., 2012). Compatible with these researchers' conceptions, our operationalization of quadratic change emphasizes the importance of students identifying constant AoC of AoC, which is the defining characteristic of quadratic relationships.

Hence, with designing for mathematical abstraction (Pratt & Noss, 2010) in mind, we 3D-designed and printed a set of manipulatives with these two objectives in mind. First, we printed four consecutive gray triangles (Figure 6, left) to represent the growing triangle at four equal-integer increases in base length. We intended for these to provide students the opportunity for sensorimotor engagement with the growing triangle's varying area and base length. Additionally, we printed five AoC blocks (one black triangle and four differently-colored trapezoids, Figure 6, middle), which represent the amount added to each gray triangle to get to the next consecutive triangle.

Theory informed the design of the triangles and AoC blocks as we attributed them with particular affordances for likely manipulations that we hypothesized would enable students to construct and compare quantities. Specifically, we conjectured that the gray triangles may invite students to stack them on top of each other to construct quantities for the triangle's changing base length and area. We also conjectured that the AoC pieces might (1) support students in constructing AoC of area as a quantity unto itself, and (2) provoke them to stack one AoC piece on top of the next one in order to construct constant AoC of AoC non-numerically by observing the AoC of AoC as congruent rhombi (as seen in Figure 6, right).

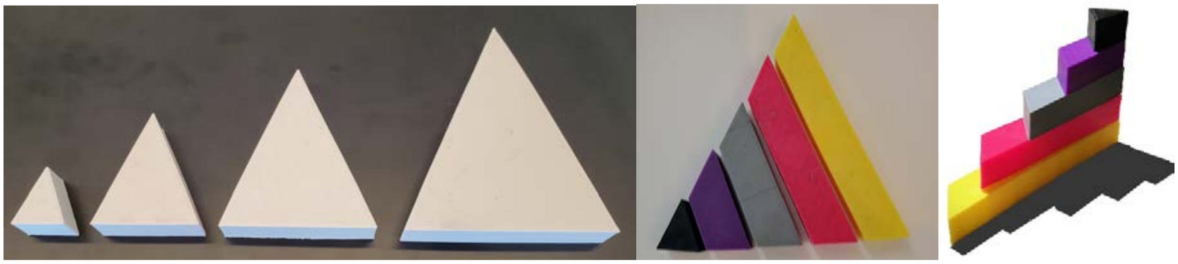


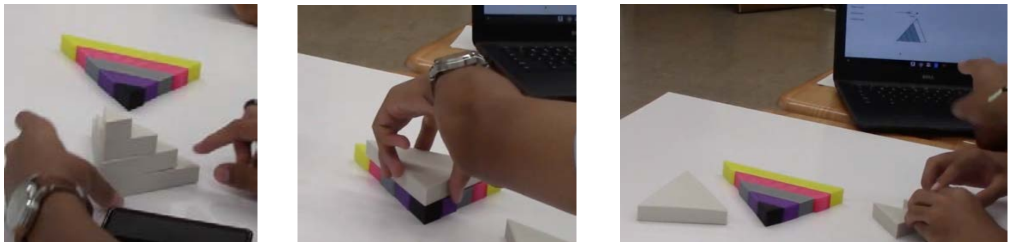
Figure 6: The 3D-printed manipulatives for the *Growing Triangle Task*

2.2.1. Using the manipulatives and the applet to construct a relationship between area and base length. We implemented the *Growing Triangle Task* during a whole-class teaching experiment with 8th grade students. All students worked in small groups while we video and audio-recorded a trio of students (Neil, Aaron, and Nigel). We highlight the interplay of the students' use of the manipulatives and applet throughout their activity.

At the outset of the *Growing Triangle Task*, the TR distributed the manipulatives while the students had the applet open. As we conjectured, the students used the manipulatives to construct and compare quantities. They immediately organized the AoC pieces to match the image on the applet showing the AoC (seen in Figure 7c). Additionally, Neil stacked the gray triangles to identify the constant amount of increase in these triangles' base lengths (Figure 7a), and placed each of the gray triangles on top of the triangle comprised of AoC pieces to describe a relationship between the different manipulatives. For instance, as Neil laid the first gray triangle on top of the black triangle, he explained, "This is that and then this one [*placing the second gray triangle on top of the black and purple piece*] is these two." He then overlaid the third and fourth gray triangles (Figure 7b) to confirm that each triangle's area corresponds to the sum of the areas of consecutive AoC pieces. Shortly afterward, the TR asked the students to describe what their goal was in organizing the AoC pieces in order to make sense of how the triangle was growing. Neil responded that his goal was to:

> Kinda model what that [*pointing to the computer screen with the AoC pieces shown, Figure 7c*] is showing. So, like this one [*pointing to the black triangle*] is like the first one, then this together [*putting fingers on black and purple AoC pieces*] would be the second one. Just basically showing how much this thing grows by each time [*motioning over the next three AoC pieces*].

Next, the TR revoiced Neil's description and asked the students to compare the relative size of the AoC pieces. As the TR motioned as if to demonstrate that he was starting with the black triangle and adding consecutive AoC pieces, he asked the students how the collective area was changing. All three students indicated the area was increasing by “more” for these consecutive changes in base length (MA3).



(a) (b) (c)

Figure 7. The students using the manipulatives to (a) determine changes in side length, (b) compare the grey triangles to the AoC pieces, and (c) to describe a relationship between the manipulatives and applet.

2.2.2. Using the 3D printed manipulatives to compare AoC of AoC. When we later returned to the task with the intention of supporting students in imagining *constant* AoC of AoC, the 3D printed manipulatives again proved useful. Because the students had already identified increasing AoC of area with respect to the increasing side length, the TR asked the students to consider, “How do the amounts of change in area compare to one another?” He invited the students to “play around with the manipulatives and explore.” Neil and Nigel began to stack the trapezoidal manipulatives on top of one another as shown in Figure 8a. The following conversation ensued:

Neil: [Places the yellow trapezoid on the bottom and then places the pink trapezoid on top] Ooh! Look, look, it’s a quadrilateral [*pointing to the piece at the end of yellow trapezoid not covered by the pink trapezoid, as seen in Figure 8b*]

Aaron: Mmhmm. [*Agreeing*]

Neil: Right, look, if we do it again. [Nigel places the gray trapezoid on top of the pink trapezoid.] Same size [*pointing to quadrilaterals created by the difference in the yellow and pink trapezoids, and pink and gray trapezoids, to indicate that these amounts are the “same size”*]. Put that one on [*referring to the purple trapezoid*]. [Nigel places the purple trapezoid on top of the stack.] Same size [*referring to the quadrilateral formed by the difference in purple and gray trapezoids*].

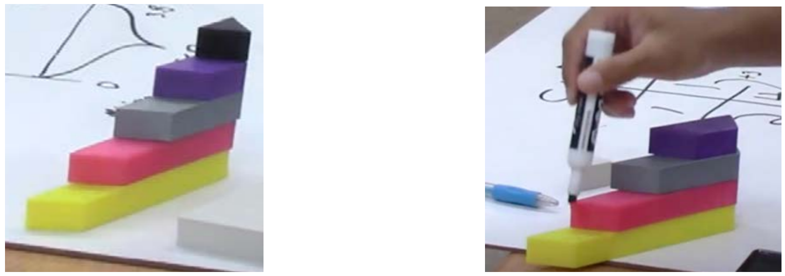
TR: So, what did you notice?

Neil: It’s like, this quadrilateral [*points to the quadrilateral shown by the differences in area, as seen in Figure 8b*] keeps going, I guess, it’s added on to that [*pointing to each layer of the stack of trapezoids*].

TR: That, that piece we’re adding on to the amounts of change is always the same? [*Nigel nods in agreement as TR speaks.*]

Neil: Yeah. [*as TR is completing his remark*]

Through this conversation, Neil characterizes the amounts being added to each of the trapezoidal AoC blocks are equal, that is, he identifies a physical representation of (non-numeric) constant AoC of AoC.



(a) (b)

Figure 8: Neil and Nigel use the manipulatives to identify a constant second difference.

Additionally, we conjecture Nigel and Aaron also understood that the relationship between area and side length exhibited constant AoC of AoC based on their activity here (e.g., adding consecutive trapezoids to the stack) and their later activity which entailed them leveraging constant AoC of AoC. In that case, we presented them with tasks representing a relationship with constant AoC of AoC between volume and length for a hypothetical situation and provided them with three values of length and volume. Aaron and Nigel calculated both the AoC in volume and the AoC of AoC in volume. They then used the constant AoC of AoC to determine the next volume value. We infer from these actions that the students leveraged a meaning for constant AoC of AoC, which they first constructed using the manipulatives, to calculate values of new points in a given relationship.

2.2.3. Reflecting on the *Growing Triangle Task*. We designed the 3D printed triangles and AoC blocks with affordances that we hypothesized would enable students to coordinate their perceptual elements in order to construct quantities for the triangle's changing base length, area, AoC in area, and AoC of AoC of area. We highlight how, through their sensorimotor engagement with the 3D-printed manipulatives, the students were able to construct AoC as a quantity unto itself in a way that supported them in comparing the relative changes in AoC to identify constant AoC of AoC, the defining characteristic of quadratic growth. Hence, the 3D-printed manipulatives supported us in making salient another aspect of this covariational relationship, the constant AoC of AoC, which proved useful in supporting students developing meanings for quadratic change (see Authors, 2020).

3. Discussion

The integration of 3D design and printing was part of a responsive design approach taken by the research team to resolve issues with students' construction of quantities in the *Cone Task*. However, once we were aware of the transformative nature of 3D printing, it was easy for us to leverage this new technology to support students in developing more productive meanings for other mathematical ideas (e.g., quadratic change). In our discussion, we first describe how 3D design tied into the theoretical perspectives we outlined at the outset of the paper. We conclude with implications for learning and teaching across STEM education.

Our presentation of the task design and sequence highlights ways in which theories of quantitative and covariational reasoning intersected with the theory of designing for mathematical abstraction to inform the design of a viable task sequence. By responsively designing and 3D printing manipulatives for both tasks with our theoretical framings in mind, we were able to support students in conceiving of quantities and changes in quantities so that through their representational activity, they could then coordinate how these quantities covary.

Particularly, we highlight how the dynamic applets and 3D-printed manipulatives served complementary purposes in supporting students' quantitative and covariational reasoning. Specifically, the applets supported students in conceiving of smoothly changing quantities (Castillo-Garsow et al., 2013), which was critical to their situational understanding and evident in their later activity as they graphed the relationships between these quantities. However, we were mindful of the fact that simply conceiving of a smoothly changing phenomena may not prepare students to engage in the mental actions described by Carlson et al. (2002). Although each applet was designed with equal changes of one quantity in mind, it was the 3D-printed manipulatives in the *Cone Task* that were critical for students to discern quantities and non-numeric changes in those quantities. Additionally, the 3D-printed manipulatives enabled students' constructions of non-numeric constant AoC of AoC, which became a defining characteristic of quadratic growth for those students.

4. Conclusions

4.1. Implications for Research in STEM Education

Specific to quantitative reasoning, we emphasize that students constructing and reasoning about quantities is non-trivial and that the particular constructions the students in this study accomplished are remarkable. We highlight the extent to which students made viable claims regarding the situation presented in the *Cone Task*, even though at times they were reasoning about a quantity (i.e., volume) that was different than what we had intended (i.e., outer surface area). If we had only assessed students' understandings based on their conclusions (e.g., surface area increases by more for equal changes of height), we would not have been able to discern this disconnect. Hence, we re-emphasize researchers' calls for STEM educators and researchers to take students' construction of quantity more seriously (Moore & Carlson, 2012; Thompson 2008; Steffe 1991).

Whereas students' construction of specific quantities in this study were often secondary to their mathematical activity, developing robust meanings for complex quantities is critical across STEM fields. For instance, research (e.g., Reif & Allen, 1992; Tasar, 2010) has described persistent difficulties students experience developing meanings for relationships between quantities such as velocity, acceleration, and force, which are critical for physics and engineering majors. Thus, we contend that our findings have implications for research beyond mathematics and across STEM fields that explores students' engagement with multiple representations of phenomena as they construct quantities essential to the domain. A research agenda such as this one may yield more viable ways of supporting STEM students in constructing the target quantities researchers or teachers intend, with new access to 3D printing being one potentially fruitful pathway to explore.

4.2. Implications for Teaching and Learning

As we highlight in this paper, a responsive approach to design via new access to 3D printing can be one way to support students in reasoning about researchers' target quantities. In addition, we would make a similar argument in the context of teaching. To illustrate this point, teachers across STEM fields could take a responsive approach to designing manipulatives for use in their own teaching in ways that resonate with their noticings (Sherin et al., 2011) of students' engagement with concepts the teachers anticipate students will struggle to understand. Authors (2017, 2019) provide some examples of prospective elementary teachers designing manipulatives for students' engagement with concepts including fractions, time, and geometry. We can also imagine opportunities for responsive design in other STEM disciplines, such as in science education (Novak & Wisdom, 2018) and technology education (Verner & Merksamer, 2015), where research has begun to explore the effects of 3D design experiences in teacher preparation. Hence, we intend this report to provide some insights into ways new access to 3D printing can influence STEM teaching and learning by concretizing abstract quantities in physical, manipulatable, representations.

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