Possibilities for Making in Mathematics Education: Two Case Studies for Teacher Knowledge

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Abstract

The current qualitative study investigates the use of teacher knowledge during a Making experience in teacher education. A course project focused on Making a manipulative for a child's mathematical learning, and interviewing a child with that manipulative, provides the setting for the researchers to examine how two practicing and prospective K-6 mathematics teachers (PMTs) invoked pedagogical content knowledge (PCK) during these processes. Research questions focused on the evolving interactions between the PMTs' content knowledge, pedagogical knowledge and PCK, and describing how these interactions helped us distinguish a "reform-minded" development and demonstration of PCK. Our study found that the Making experience provided a learning setting for the PMTs to embed their knowledge into the design of their tools. Moreover, this process elicited distinct facets of content and pedagogical knowledge and their integration during authentic teacher learning moments, which supported reform-minded practices like reasoning mathematically or persisting through the "messiness" of teaching and learning mathematics. These findings suggest the promise of a Making experience for supporting knowledge development within mathematics teacher preparation.

Keywords: mathematics teacher knowledge, Making, 3D designing and printing, mathematical manipulatives

Introduction

In this article, we describe a project that was undertaken in a teacher education class for practicing and prospective, K-6 mathematics teachers (PMTs). In an attempt to expose the PMTs to a classroom experience that would combine selected mathematical and pedagogical features involved in teaching children, the teacher educator (TE) incorporated an ongoing *Making* project (Halverson & Sheridan, 2014) into the PMTs' course work. In particular, the TE tasked the PMTs with designing and 3D printing a manipulative to support a child's learning of mathematics. In addition to Making the manipulative, the PMTs would also share their tool with a child in an interview setting with the aim of developing their understanding of the child's mathematical thinking using the manipulative. The project's promise for characterizing, describing, and understanding the PMTs' knowledge for teaching informs this descriptive study, as well as its framework, methodology, outcomes and discussion. The same promise suggests the possibilities of Making for mathematical learning in teacher education, and for supporting prospective teachers in articulating thinking that reflects the complexities and nuances in their knowledge during Making activities.

Background and Underlying Philosophies

We (the two authors) and the TE are currently working together on a larger project entitled *Teachers Making for Mathematical Learning* (TMML, Blinded) that stemmed from this paper's initial study. We decided to write about the initial project, which we call our Pilot study, to introduce the teacher education community to the starting point of TMML.

This paper's experiment took place in the spring of 2017 in a graduate course entitled Math 577: Mathematics Education in the Elementary Schools. Math 577 is a content class that is taken by PMTs who are pursuing graduate level coursework toward their preparation or development in the teaching of K-6 mathematics. In the experiences of teacher educators who have taught courses such as Math 577 (which includes this study's TE and one author), PMTs typically come to this class believing that mathematics activities are (mostly) restricted to algorithms that lead to pre-determined and single solutions. They convey the traditional views of mathematics as a discipline unconnected to others, with little or no relevance outside of school. In general, students who hold such views do not understand mathematical solutions or why they work—only *that* they work (and this suffices). This thinking can lead to the repetition of memorize-able steps as the primary activity associated with the teaching, learning and doing of mathematics (see Association of Mathematics Teacher Educators 2013; Ball 1990; Lampert, 1990; Ma, 1999; McDiarmid, Ball & Anderson, 1989; Thompson, 1984).

One of the many challenges facing a teacher educator whose students hold to such views lies in compelling students to confront this thinking and consider *reform-minded* approaches to doing mathematics. For PMTs, this admits the possibility of connections between mathematical topics and between mathematics and other disciplines. A mathematics problem can now be *open-ended* or one "for which the solution method is not known in advance" (NCTM, 2000; p. 52). Problems can have solutions that are not immediately obvious, multiple solutions, or solutions that vary under different conditions. Mathematical activities can include "using diagrams, looking for patterns, listing all possibilities, trying special values or cases, working backward, guessing and checking, creating an equivalent problem, and creating a simpler problem" (National Council of Teachers of Mathematics, 2000; p. 54). Tools to support these activities can include manipulatives and technology (National Council of Teachers of Mathematics, 2014), which can encourage student thinking, and provide "rich environments for learning mathematical reasoning" (NCTM, 2000; p.58).

The Math 577 TE supported the PMTs in participating in such reform-minded practices. He designed classroom activities based in reasoning and struggling with open-ended mathematics problems; he revisited familiar topics (like fractions) in novel settings to expose PMTs to layers of thinking they may not have studied as students; and he incorporated videos into his lessons, of children reasoning and problem solving during interview and classroom settings. This backdrop was important in supporting the Making activity that he incorporated into the course. In the current study, we define Making as *the process of designing, building and innovating with tools and materials to solve practical problems* (Halverson & Sheridan, 2014). By inviting the PMTs to design a manipulative for a child's learning of a mathematics topic and sharing that manipulative with a child during an interview, the TE anticipated the PMTs could summon, use and learn facets of reform-minded practices in mathematics. For example, the task of conceptualizing and designing a manipulative is open-ended, admits the possibility of more than one solution, and compels a deep understanding of the mathematics topic intended for a child's learning. This thinking echoes Papert's *constructionism*, which is often credited as the theory underlying Making and which supports learning mathematics "through participation in other activities than the math itself" (Papert, 1993; p. 145). The hope, then, was that the project would give the PMTs an opportunity to challenge, explore, and extend their mathematical thinking.

As part of the course project, the TE also gave written assignments for PMTs to reflect on the different phases of their experiences in designing and sharing their tool. As we (the authors) reviewed these assignments to prepare for the larger TMML study, we realized that some of the PMTs' writing revealed a deep and connected understanding of mathematics not typically documented in teacher education research. We sought to conceptualize this observation to share with the teacher education community by delving more deeply into the research on Making and on knowledge for teaching, and how these can interact in a teacher learning setting.

Framework: Making and Teacher Knowledge

As noted, we draw on the conception of Making as the process of designing, building, and innovating with tools and materials to solve practical problems (Halverson & Sheridan, 2014). This definition can be connected to our project by elaborating its components: the *practical problem* presented to the PMTs was that of creating a hand-held manipulative to support a child's learning with a mathematics topic. The *tools and materials* included a web-based, geometry design software called Tinkercad (Autodesk, 2016) that allows users to group and cut pre-made, three dimensional objects to produce new forms (see Fig. 1), as well as the 3D printer for printing the resulting shape.

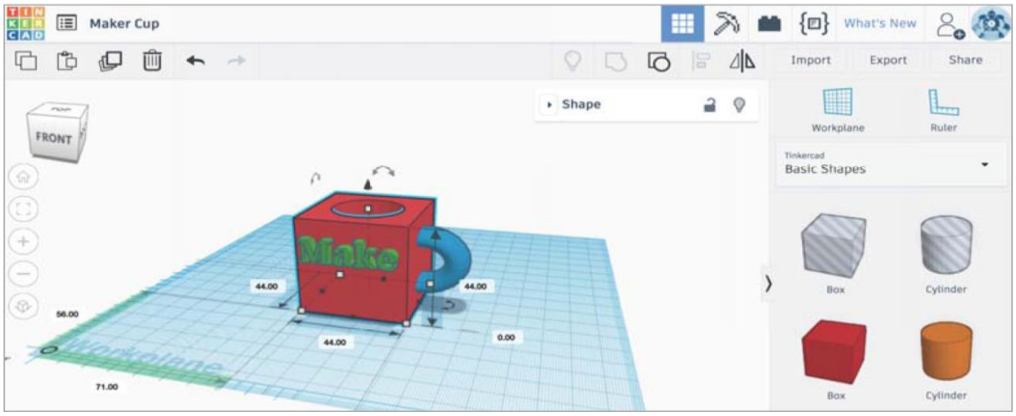


Fig. 1. The Tinkercad Workspace

The process of *designing* took place, in part, upon this workspace and included the purposeful imagining and planning that preceded working with this software and while interacting with it. In sum, our approach called upon PMTs to "actively engage in inquiry, research and design" so that they could make "tangible, meaningful artifacts" that represent "the end products of the learning process" (Koehler & Mishra, 2005; p. 135).

One of the TE's objectives was for the PMTs' Making of a manipulative to be informed by a desire to deepen or enrich a child's understanding of a mathematics topic. Because an actual child would use the manipulative to engage with the topic, this brought an element of concrete specificity to the PMTs' work. Such considerations for mathematics content, a real child's reasoning and struggles, and tools to support a child's learning elicited (for us) Shulman's (1986) *pedagogical content knowledge* (PCK) for teachers. Unique to the teaching profession, PCK is that special domain of knowledge that captures how teachers blend knowledge of content and pedagogy "into an understanding of how particular aspects of subject matter are organized, adapted and represented for instruction" (Mishra & Koehler, 2006; p. 1021).

PCK can be thought of as having a component structure consisting of content and pedagogical pieces (Mishra & Koehler, 2006). The content portion of PCK is knowledge of the actual subject matter being learned and taught. For Shulman (1986), content knowledge elicits an understanding of a subject's facts, concepts, theories and principles as well as knowledge that informs how frameworks can be used to organize and connect a discipline's ideas. It also encompasses how reasoning happens in a discipline and how legitimacy is established through this reasoning (Mishra & Koehler, 2006; Schwab, 1978; Shulman, 1987). The pedagogical portion of PCK is concerned with matters that transcend content, like issues of organization and management (Shulman, 1987). It includes "knowledge about techniques or methods to be used in the classroom; the nature of the target audience; and strategies for evaluating student understanding." (Mishra & Koehler, 2006; p. 1026). As PCK integrates these considerations into one category, it recognizes the necessary relationship between a teacher's knowledge of pedagogy and content, and that teachers are continually negotiating this relationship within their practice. The importance of enhancing PCK and its defining features during teacher preparation has been well documented by the teacher education community (AMTE, 2013; Borko & Livingston, 1989, 1990; Darling-Hammond, 2006; Shulman, 1986).

As we revisited Shulman's (1986) oft-cited passage, the connections between PCK's utility within a Making experience deepened. In Shulman's passage (italics added), for a subject's most regularly taught topics, he details the features of PCK as including:

> the most *useful forms of representation* of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations, in a word, the ways of representing and formulating the subject that make it *comprehensible to others*. . . pedagogical content knowledge also includes an *understanding of what makes the learning of specific topics easy or difficult*: the

conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics or lessons. (p. 9)

We posit that the manipulative a PMT creates and the design decisions that go into its Making can provide a rich resource for understanding what the PMT values as one of the "useful forms of representation" for a mathematical idea. Because the PMT would be engaging a child in investigating and understanding the idea, the PMT's associated thinking also reflects their effort to make the mathematical concept "comprehensible to others." And consideration of how using a manipulative could address a child's misconceptions or limited views about a topic could also reveal a PMT's "understanding of what makes the learning of specific topics easy or difficult."

With PCK as a framing device, we utilized its component structure to examine how the process of design interacts with the process of transforming pedagogical and content knowledge for teaching. Pratt and Noss (2010) offer support for examining this interaction by recognizing that the intentions and understandings of the designer impact the design of a tool, moving from left to right in Fig. 2, below. This suggests that the PMTs' developing PCK can get reflected in the design of their tool. But the process of design introduces an added (reciprocal) opportunity for investigating knowledge transformation if we consider that the designer's "understanding reflects the design of the tool" (Pratt & Noss, 2010; p. 96), moving from right to left in Fig. 2. This admits the recognition that as the design experience involves continual renegotiations of knowledge, this can impact PCK.

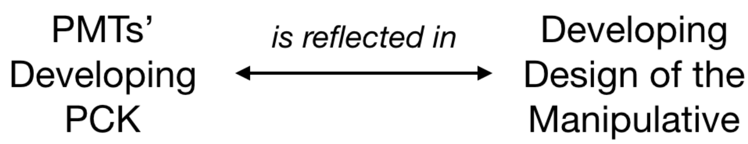


Fig. 2. Relating PCK and Making a Manipulative

More specifically, the left side of our model, *PMTs' Developing PCK*, recognizes the PMT's negotiation of content and pedagogy during the design process, or how the PMT "interprets the subject matter and finds different ways to represent it and make it accessible to learners" (Mishra & Koehler, 2006; p. 1021) through their tool. The right side, entitled *Developing Design of the Manipulative*, captures how the design of the manipulative evolves. We intend this framework to help us utilize both the component structure of PCK (pedagogical, content, and pedagogical content knowledge), and the transformations that arise when these components are blended during the designing and sharing of a manipulative. Investigating and describing how each part of the model is reflected in the other provides an opportunity to consider the reciprocity between the evolving artifact and the uses of PCK in the artifact's development. This framework echoes more recent thinking on teacher knowledge that recognizes the "significance of the context in which knowledge comes into being" (Scheiner, Montes, Godino, Carillo & Pino-Fan, p. 162) and encourages attention to "transformations that arise when knowledge elements are blended" (Scheiner, et al., p. 165). Guided by images of reform-minded teaching, the activity of Making, and our framework, the following research questions emerged:

- *How can the process of designing a manipulative for a child's learning of mathematics help us understand the evolving interactions between a PMT's content knowledge, pedagogical knowledge and PCK?*
- *How does examining and describing this process help us distinguish a "reform-minded" development and demonstration of PCK?*

Methodology

The current study focuses on providing rich descriptions of the relationships between PCK and designing. This focus suggests a qualitative approach to data collection, sampling, analysis and presentation (Patton, 2015). The study's Math 577 class contained twenty-six students who organized themselves into twentyone *working groups* for the project (depending on a desire to work alone or with a partner). The TE created the project's written assignments as devices for the PMTs to reflect on their work of designing manipulatives. As part of this study's follow up, the TE clarified through an email that his objective was to support reflection, and not to generate data for any kind of study. This positions our work as *naturalistic*, one whose situation was neither manipulated nor controlled and which was open to emerging circumstances (Patton, 2015). Thus, the class proceeded as the class would have proceeded with or without the current reporting.

Data Collection

The raw data for this study stem from the class's assigned papers and evolving manipulative designs. The TE guided the PMTs in their writing by presenting the following project description:

> ***Making for Learning Project***
>
> *The purpose of this project is for you to 3D design and print a new physical tool (or "manipulative") that can be used in teaching a mathematical idea. The design of this tool and a corresponding task will reflect 1) your knowledge of what it means to do mathematics and how we learn with physical tools, 2) your knowledge of elementary-level mathematics content, and 3) your perspective on pedagogy and curriculum in mathematics education. This project has three components:*
>
> 1. *Project Idea Assignment*
> 2. *Project Rationale Assignment*
> 3. *Final Paper and Design Show*

These writing assignments were coupled with additional support and activities, and are organized below into three phases each containing two tasks (see Fig. 3):

	Task 1	Task 2
Phase 1	Tinkercad Design Time	Project Idea Writing Assignment
Phase 2	Clinical Interviews With Child	Project Rationale
Phase 3	Final Design Show	Final Paper Writing Assignment

Figure 3. Project Phases

Phase 1's tasks included Tinkercad training and the Project Idea Writing Assignment. This phase's in-class Tinkercad training and designing continued over the course of six class meetings during which the TE worked with PMTs to learn the software's features and apply them in pursuing design ideas within the software's platform. These meetings included time to collaborate and design a manipulative both on, and outside of, the Tinkercad workspace. The purpose of the Project Idea assignment was to get the PMTs thinking about a manipulative they might want to design, and to serve as a baseline for their thinking in the Making process. Phase 1's tasks were purposefully juxtaposed so that the PMTs would have an imagined tool to work toward, but also, so that they could experience how their developing knowledge with Tinkercad would enable and constrain their ability to make that manipulative.

As the semester went on, the PMTs continued working on Phase 1's activity of designing into Phase 2. In Phase 2, the TE also introduced the Clinical Interviews with a child and assigned a Project Rationale Paper. For the interviews, the PMTs selected a child who was in elementary school and who could participate in all three of the project's interviews. In this phase's first two interviews, the PMTs posed mathematical task(s) to their child and probed the child's thinking on his or her approaches. If the task involved working with an existing mainstream manipulative, the PMT could additionally observe how the child used the manipulative to assist his or her learning and thinking. For the Project Rationale, PMTs were asked to think more deeply and write about how they expected *their* manipulative to work from a learning standpoint with a child. Again, a purposeful juxtaposition of watching an actual child learn using mainstream manipulatives, while imagining how that child might learn from *their* manipulative advanced the PMTs' thinking toward the final phase.

During Phase 3, the PMTs continued working toward completing and printing out their manipulatives. In these last weeks of the semester, the PMTs conducted their final interview with the child, posing mathematical task(s) and probing how the child could use their manipulative to reason about the task(s). With the final interview behind them, the final phase's design show and writing assignment provided an opportunity for the PMTs to reflect back on their experience. In the Final Paper, the TE instructed the PMTs to write about their project manipulative design and rationale, factors that influenced the evolving manipulative design, the task(s) posed during the final clinical interview, and the findings from the interview. The PMTs also were instructed to look back on what they had learned about design and about working with their child, and what they would change about the interview or design experience. The Design Show involved creating a display (a poster with their project manipulative) for presentation to their peers and faculty.

Purposeful Sampling and Data Analysis

The Project Idea, Project Rationale, and Final Paper comprise this study's written data. Screenshots and Tinkercad files of the manipulatives along with screenshots of the posters (from the Final Design Show) comprise this study's pictorial data. Our approach to data analysis included an initial individual round in which we examined the data separately, followed by a collaborative round in which we came together to share findings. We then repeated this process for a total of four rounds of data analysis. For us, these rounds presented a device for honoring and presenting (to one another) our respective interpretations of the data (Patton, 2015). That is, our independent work followed by discussion were treated as settings where "multiple analysts might still discuss what they see in the data, share insights, and consider what *emerges* (italics added) from their different perspectives" (Patton, 2015; p. 667). This point of view authorizes researchers "to trust themselves and their judgments" and "to defend their interpretations and analysis" so that the richness and creativity of qualitative work are not compromised (Morse, 1997, cited from Patton, 2015; p. 667).

In our first round of analysis, we independently read through all the PMTs' papers and examined screenshots of the PMTs' tools, using the literature on teacher knowledge (see Mishra & Koeher, 2006; Shulman, 1986) as a lens for orientation. After coming back together, we communicated our respective observations and articulated the same two working groups as standing out. In particular, we were drawn to these PMTs' uses of knowledge of mathematics in describing their design experiences. For example, we realized these groups shared views about the limitations of rote learning and mnemonics in mathematics, and acted on these through the design experience. We also noted these PMTs making authentic connections between mathematics and the tools they were designing, and within the interview contexts that they arranged. This prompted us to invoke a *case study approach*, defining a case as a working group of PMTs (Creswell, 2007). Drawing from the methodological stance for purposeful sampling (Creswell, 2007), and harnessing the case study's virtue for "evoking images of the possible" (Shulman, 2004; p. 147), we opted to focus the current paper on these two cases.

We worked independently (again) to more critically analyze the identified cases so that we could fully flush out the PMTs' emerging ideas. And we planned to reconvene to compare narratives reflecting knowledge use during the design experience. As intended, this approach helped us write a case study for each of the two working groups that enhanced our collective understanding of the PMTs' uses of PCK and its component features during their design experiences.

Data Presentation

Our two case studies are titled by the pseudonyms we used for the associated PMTs and by the names of their tools. Thus, we present the results of our analysis for Avery and her *Even Number Tool*, and for Casey and Mia (who worked together) on their *Minute Minis*. For each case, we organize our results around the PMTs' designing and interviewing activities. The presentation of our results is typical of qualitative presentation in its use of rich textual data but also includes screenshots of the PMTs' manipulatives and their Tinkercad files. We view the pictures as reflections of the PMTs' evolving knowledge and as enhancing the meaning in the PMTs' papers. We encourage the reader to carefully consider both the text and figures, and their relationships to each other, as they read the upcoming sections.

Results

Avery and Designing the Even Number Tool

Avery is the designer of our first tool. She began her design experiment hoping to create a math puzzle, but by the time of her Project Rationale, she had honed in on something else: she wanted to create a manipulative that would help students distinguish even and odd numbers. It was important to her that the distinction not rely on rote memorization of lists of numbers:

> For my 3-D design project, I designed a manipulative that I believe will help students learn what an even number is. As we discussed, it is better to learn what something is and understand it, instead of simply memorizing the facts. Therefore, for my manipulative to help learn what even numbers are, I scratched my puzzle idea.

Avery reflected on what it would mean for her project to fail, writing, "the project would fail if students memorized the rule for even numbers, instead of using the design to explore and learn." Avery's definition of an *even number* was "any integer . . . that can be divided exactly by 2." In her Rationale, she describes her overall tool having two parts: (1) a *container with two compartments* and (2) *objects* to place into each compartment. She intends a child to take a certain number of objects (say, 4 or 5) to represent the number whose parity was being tested and distribute the objects into the compartments, one at a time, until none remain. The balance or imbalance in the compartments was intended to help a child reason about a number's parity. Avery's design begins to unfold as she creates pyramids in Tinkercad for the tool's objects. The pyramid's bases are regular polygons with 4, 5, 6, 7, or 8 edges. Her idea is to strategically cut each pyramid into "pieces" (see Fig. 4) so that the compartments can be used to separate the pieces of a given pyramid, "one at a time, into the two sides of the container." Because of the way the pyramids are designed and cut, the number of edges reflects the parity of the number being tested.

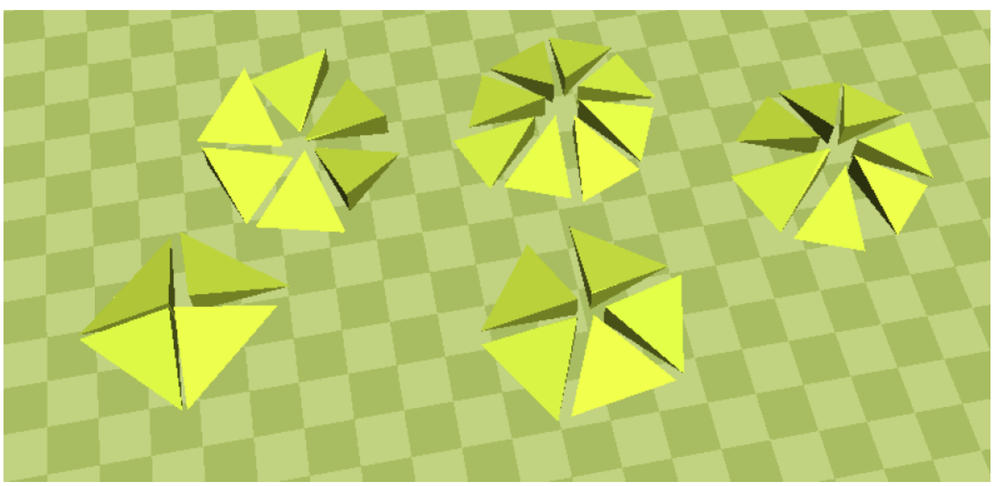
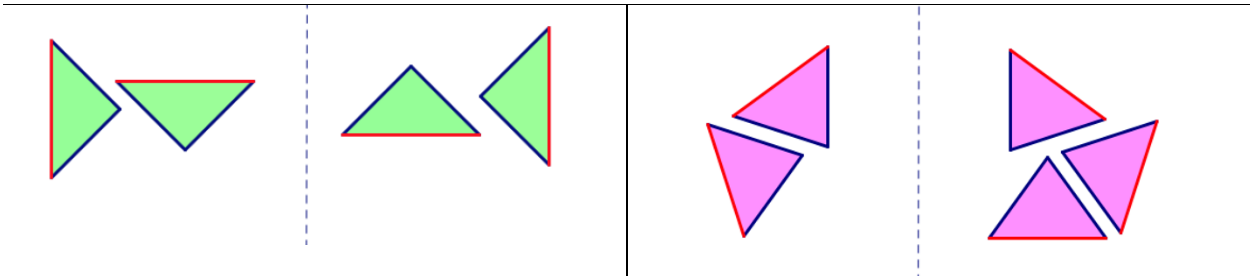


Fig. 4. Avery's Initial Design for the Objects
(STL File for Even Number Tool from Tinkercad)

For example, to test the parity of the number 4, a child could allocate the 4 pieces of the square pyramid one at a time into each compartment of her container until there were none left (see Fig. 5). For the number 4, the compartments would be balanced, and a child could reason that 4 is even since the 4 pieces can be divided exactly into the 2 compartments. For the number 5, a child would distribute the pieces of the pentagonal pyramid, compartments would be imbalanced, and a child could reason that 5 is odd since the 5 pieces cannot be divided evenly into the 2 compartments (see Fig. 5).



For the square pyramid: testing number 4, 2 “edges” on each side	For the pentagon pyramid: testing number 5, 2 “edges” on one side and 3 “edges” on the other side

Fig. 5. Even and Odd “Numbers” Using the Pyramids

Avery articulates additional design features by purposefully including pyramids with either even- or odd- numbered sides. Her intention is for students to “really have to think and not assume that a number is even based on if the number of edges is a number included in this manipulative.” With this design choice, Avery appears to anticipate children associating the *even* number tool with testing *even* numbers only. In the subject of mathematics, students have had experiences in which repeating features in a list of answers or solutions betray upcoming answers or solutions. Avery reasons that if her tool’s objects contain both even- *and* odd- number edges (so that the tool can be used to test both even and odd numbers), children will be compelled to reason through each example posed.

In the Final Paper stage of writing, Avery articulates additional, undesirable aspects of rote learning as she recalls how songs or lists are used as an aide in helping students memorize even and odd numbers:

> From what I have seen in schools and from what I had experienced myself as a child, children are taught to memorize even and odd numbers with no understanding of why even numbers are even and why odd numbers are odd. They memorize songs or simply (say) that 2, 4, 6, 8, 10 are even and 1, 3, 5, 7, 9 are odd. These memorizations are what children use as confirmation to why a number is even or odd.

In addition, Avery writes about a revision to the objects for her tool after receiving classroom feedback during a Tinkercad Design Time session. This experience leads to a realization that the triangular-shaped pieces from the different pyramids would be so small and so similarly shaped that they would be easy to confuse (for example, the square and pentagonal pieces are strikingly similar). She also writes about her realization that “the shapes used are meaningless” to the concept she is trying to teach. As a result, Avery transitions to (what becomes for her) a more practical and meaningful design that uses “people” or “doll” shapes to represent the number whose parity is being investigated (see Fig. 6):

> . . . the two parts of the container represent the two halves of the whole and the people represent the individual pieces of the whole which can be split into the parts of the whole. When the two halves have the same number of people in them, it will show that the number of people can be divided by two equally. Since this design is about splitting the whole into two halves using the box and people, this would mathematically be seen as a direct modeling strategy for Partitive Division. . . . The box pre-establishes the number of groups, so the children place the people into the said groups (sic). From there, students can determine if the whole number of dolls is an even or odd number.

Fig. 6. Avery's Final Even Number Tool Design

With this reflection, Avery communicates how her knowledge of mathematics content, pedagogy and PCK inform the design and intended use of her tool: she invokes mathematics content knowledge in drawing a connection between determining a number's parity and allocating "pieces" of the number partitively into the divisor of two compartments. But her attention to placing pieces into groups to help children reason also reflects an instantiation of PCK as it focuses on "useful forms" of representing an idea to support a child's reasoning. The manipulative's evolution also is striking. Under the original design, the plan was for the child to have the number of edges for a given shape represent the number under investigation. But Avery summons pedagogical knowledge with the realization that this can get confusing, particularly if children jumble pieces of one pyramid with another. Her content knowledge also comes into play when she recognizes that the pyramid shapes are not necessarily meaningful to determining the number whose parity is being tested. For example, the number 4 can be counted out of arbitrary objects rather than attached to the sides of a square. With a box partitioned into two compartments and identical objects to distribute within these compartments, Avery's final tool reflects what she considers to be essential components for investigating a number's parity: the child is not called upon to memorize; the child can experiment with even or odd numbers; and the child can reason by invoking the partitive process of division.

Avery Shares Her Even Number Tool

Avery tested her even number tool in a clinical interview setting with a third-grade boy named Michael. She had created a worksheet that featured the numbers 4, 5, 6, 7 and 8 for the interview. She utilized this resource to explain "that an even number evenly splits into two and that an odd number does not evenly split into two." On the worksheet, the student needed to decide whether the featured number was even or odd and explain why.

Avery gave Michael the worksheet, observed that he completed it quickly, and that Michael remarked, ". . . the questions were very easy." She also observed that he did not use the manipulative for the problems asking about even numbers, but that he did use it for problems asking about odd numbers. Avery describes asking Michael about even numbers and his reply, "that they are always going to *equal the same thing*." Because she wants "to know more about his knowledge of even and odd numbers," she continues:

> . . . "what tells you a number is odd?" He then told me that it is because it doesn't equal the same thing. Then I said, "what tells you that a number is even?" His response was that a number is even if its (sic) 2, 4, 6, 8, 10, 12. This was interesting to me because I had known that most, if not all, children only memorize what even and odd numbers are. This explanation showed me that he memorized those numbers as being even. I then pushed to see if he knew more. I asked why those numbers are even. When asked why, the student said that it was because they always equal the same thing.

Avery's recollection of this interview combines important features about the design of her tool and its intended role in a child's learning. From these segments, we inferred that his language reflects the mathematical relationship between the objects in the manipulative's compartments after partitioning: if he partitions an even number of pieces, each compartment will end up holding "the same thing;" if he partitions an odd number, "it doesn't equal the same thing." Michael also observes that an even number can be characterized by its membership in a list—a description that Avery believes he memorized from a prior encounter (which is supported by his initial disregard for using the manipulative to respond to problems about even numbers). By engaging with the posed tasks using Avery's manipulative, we inferred that Michael learned to see even numbers through a more meaningful lens. His prior conception that even numbers belong to a memorized list expands to include one in which the partitive allocation of pieces into compartments generates "the same thing." In her writing, Avery recognizes the value of the manipulative in supporting the thinking process of its user. She concludes that using the manipulative to reason about a number's parity can enhance the child's understanding beyond that of mnemonics (which was one of her goals). We also note that by sharing her manipulative with a child, Avery participates in developing a child's thinking about even numbers and in fostering a way of understanding them that is mathematically meaningful.

Casey & Mia and Designing the Minute Minis Tool

Our second case study involves Casey and Mia who worked together on their manipulative. At the outset, each posed a different tool for making under the Project Idea Assignment. Casey envisioned a tool for helping young children tell time, "something similar to fraction circles but that would be labeled with the amount (*sic*) of minutes," and Mia conveyed her desire to make a tool that would "help children learn about fractions." She writes of her wish to alleviate anxiety surrounding a topic that, "for most children, when they hear the word fractions, they immediately tend to have a level of anxiety and discomfort arise in them." She calls her manipulative "fractions cakes" and realizes that there are many options for a shape that can underlie its design (like a circle, square or rectangle). With Casey's interest in the topic of telling time, Mia is ultimately drawn to a circle shape so that the design of their manipulative can be compatible with the face of a clock. This focus provides an opportunity to think about how the fractional pieces of a clock can embody the fractional ideas of time.

Casey and Mia recognize the abstract nature of time and the meaning that manipulatives can bring to learning. The meeting of these ideas leads them to make a concrete time manipulative:

> The main goal of this project is to give a concrete representation of the relationship between hours and minutes. Using manipulatives is especially important when exploring new concepts, and sense (*sic*) time is a very abstract concept, it is especially pertinent that students have something concrete to work with. . . .

As with Avery, Casey also articulates the missed opportunities that meaningless memorization can generate, but for her, this memorization is described with regard to learning time:

> Currently, most of the 2nd graders in my class can tell time to the nearest half hour, yet I am unsure of how they know how to do this. Is it just because they know that when the minute hand is pointing at the 6 I say _:30 and when it's pointing at the 12 I say _:00? Or do they have a more (*sic*) deeper understanding of time and how a clock works?

By the time of the Rationale phase, Casey and Mia's design is in place. They hold to their overall design framework which is based in an existing model of *fraction circl*es, stipulating that, "instead of labeling them (fraction circles) with a fraction, they will be labeled with minutes. For instance, a whole circle will be labeled '1 hour,' while two half circles will be labeled '30 minutes.' We will also have fraction circles for 15 and 5 minutes." (see Fig. 7)

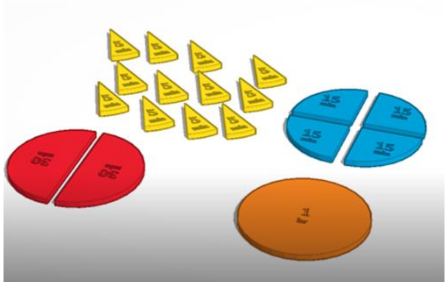


Fig. 7. Minute Minis

Casey and Mia reflect further on the value of the fractional pieces that underlie a clock face, making a connection between the clock's numbers (1, 2, 3, 4, . . 12) and the passage of time:

> . . . students would be able to use the Minute Minis directly on the face of an analog clock. This could aid in exploring the relationship between where the minute hand is pointing and the number of minutes past the hour. . . In other words, students can use the pieces to help them count, i.e. "I know that when the minute hand is pointing at the 4, it is 20 minutes past the hour because 15+5=20. I started at the 12 on the clock and put a 15 minute and a 5 minute piece to get to 4." We also wanted to ensure that we used fractions that would equate to important benchmark times (15 minutes, 30 minutes).

Casey and Mia's ideas develop further as they investigate a solution to a problem called the Bob Homework Problem: *"Bob has four homework assignments. It will take him 45 minutes to finish each assignment. How many hours of homework does Bob have in total?"* In reflecting on a possible solution, their manipulative gives them a set of options (15- and 30- minute increments) that help to inform the problem design:

> A potential way this could be modeled would be using one 30 minute piece and a 15 minute piece to show 45 minutes, then replicating this 3 other times to show 4x45. A child might then notice that they can make 2 wholes - hours in this case - using 4 of the 30 minute pieces and 1 whole/hour using the four 15 minute pieces, leading them to an answer of 3 hours.

Through their reflections, Casey and Mia describe how to utilize the geometry in an analog clock and its circular sectors to design their tool. In particular, they embed their content knowledge of fractions and area into a design that can act as a bridge between the abstract and concrete qualities of time. They also utilize some of their design's essential components through solutions to posed problems. For example, their solution to the Bob Homework problem reflects a combination of their own thinking on the manipulative and its uses (thereby reflecting their content knowledge) and a child's thinking (thereby reflecting their PCK). By giving a voice to an imaginary child's possible approaches to a problem, we witness their design in action, and the mathematical ideas underlying their manipulative's intended uses.

Casey and Mia Share Their Minute Minis Tool

Casey and Mia's interview focuses on a time conversion task with a 9-year old child named Rocco. Casey and Mia divide the interview into three phases. In the first phase, the child played and familiarized himself with the manipulative. In the second phase, Casey and Mia asked the child a mathematical question that would prepare for the next and final phase. In this last phase, the child uses the manipulative toward solving a posed task.

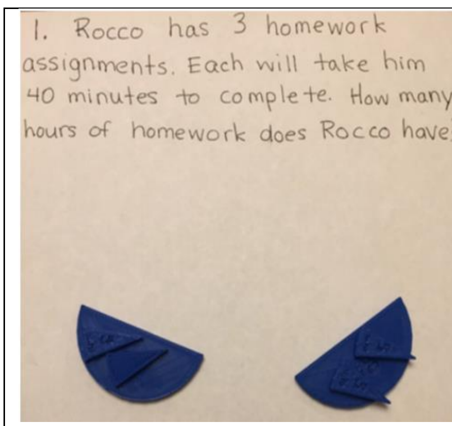
Casey and Mia write that the first two phases of the interview go "as planned": Rocco begins by playing with the Minute Minis during the first phase. During the second phase, Rocco was asked to investigate how many 30-, 15- and 5- minute pieces make up one hour. The PMTs write that this was designed to set up a foundation for Rocco's thinking about time conversions:

The second phase was to get the child familiar with the concept of creating hours out of minutes. The child would be asked the following: *How many 30 minute pieces make an hour? How many 15? How many 5?* Recognizing how many of each type of Minute Mini create an hour would hopefully set up the child for success when solving time conversion problems; they would already be familiar with how to combine the pieces into wholes.

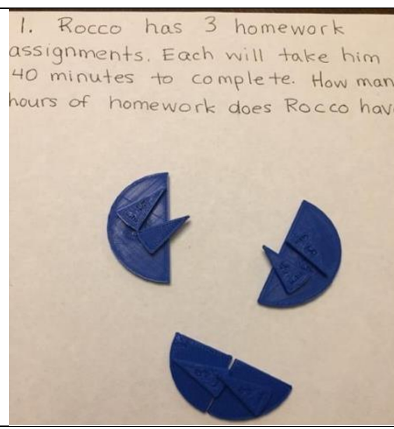
In the third phase, Rocco is given a time conversion problem that was personalized for him and based on the Bob Homework Problem: "*Rocco has 3 homework assignments. Each will take him 40 minutes to complete. How many hours of homework does Rocco have*?" Casey and Mia report that Rocco immediately volunteers "120 minutes" as a possible response (in much the same way that Michael responded quickly to Avery's posed problems). From prior interviews with Rocco, Casey and Mia recall thinking how Rocco prides himself on mental math and speed, and suggest to him that he slow down and read the problem again.

After re-reading the problem, Rocco replies, "Oh, I can't do that." Because the problem statement requests *hours* (and not minutes), the PMTs conjecture from this response that Rocco is struggling with the conversion part of the solution. They encourage Rocco to solve the problem using their manipulative. This prompt generates an opportunity for them to see how their tool can be used to navigate an upcoming obstacle in Rocco's learning, and to discern features of his thinking about how to solve the problem:

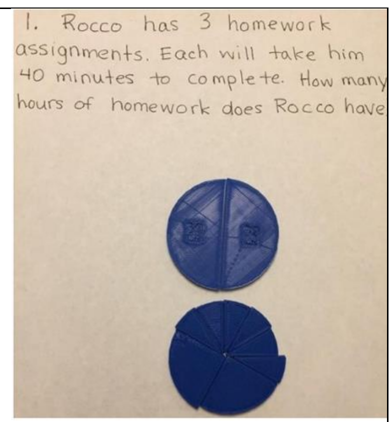
First, Rocco made 2 groups of 40 minutes using one 30-minute piece and two 5 minute pieces. To make the next 40-minute piece, Rocco paused. He asked for another 30-minute piece and when he found out there were no more, he assumed he could not finish the problem. Once prompted to see if he could make it another way, Rocco figured out that he could use two 15 minute pieces to make 30 minutes and was able to complete his third group of 40 minutes. Without prompting, Rocco started to move the pieces together to create an hour, working with the largest pieces first, then adding in the 5 minute pieces. Rocco came up with the correct answer, 2 hours. (see Fig. 8)



Rocco makes two, "40-minute representations"



Rocco uses two, 15-minute pieces to help make the third 40-minute representation



Rocco makes two hours: one hour with larger pieces, one with smaller pieces

Fig. 8. Casey and Mia's Screenshots of Rocco's Work with Minute Minis

In the reflection above, when Rocco runs out of 30- minute pieces, Casey and Mia guide him into constructing an equivalent representation (and he opts for two, 15-minute pieces to make 30 minutes). Their reflection further instantiates their tool's possibilities, namely, providing children with a concrete means to reason about features of time. Rocco's physical interactions with the fraction circles become a tangible manifestation of his mathematical understandings. That is, Rocco is able to understand mathematical concepts through physically interacting with the tool when he uses the larger pieces to create the first hour, and then the smaller pieces to create the second hour (see Fig. 8). His strategies bring the fractional ideas embedded in the tool to life and afford Casey and Mia a view into Rocco's problem-solving approaches, unveiling how his thinking and the tool interact to devise a strategy and ultimately a solution. Rocco's mathematical activity stands in contrast to his initial response, which contained only an answer presumably based in repeated addition or multiplication. Through the interview, Casey and Mia instantiate their collective knowledge of how teaching *and* learning can work in interaction with manipulatives: they use the challenge presented to them to prompt Rocco into persisting with the problem despite his perceived obstacle. In turn, they experience how sharing their tool with a child can transform potentially limiting, speedy mental math into an approach infused with deeper mathematical and conceptual meaning.

Discussion

The first question for our study was, *How can the process of designing a manipulative for a child's learning of mathematics help us understand the evolving interactions between a PMT's content knowledge, pedagogical knowledge and PCK?* We remind the reader that our theoretical stance admits the interplay between a PMT's evolving PCK (including its component structure) and the evolving design of a PMT's tool (see Fig. 2 above). For Avery, a pedagogical recognition for better organization and presentation of learning materials (namely, the different pyramids) interacts with her content realization that the concept of a natural number need not be anchored to a fixed geometric shape. This interaction frees Avery to abandon the pyramids and invoke PCK's attention to a more useful way of illustrating the number being tested (Shulman, 1986). For Avery, the "people" shapes represent the number with a cleaner, more accessible, and more straightforward design than her original pyramids.

Another interaction among Avery's knowledge components arises when she identifies her design as reflecting what "would mathematically be seen as a direct modeling strategy for Partitive Division." That is, she relays an understanding that the physical act of dividing partitively (content knowledge), made possible by her tool (pedagogical knowledge), can be utilized to help a child reason about a number's parity (pedagogical content knowledge). It is unclear to us whether these PCK interactions impacted the tool's design (going left to right in Fig. 2's model), or the exercise of reflecting on the design experience impacted the PCK interactions (going from right to left in Fig. 2's model). What *is* clear is *how* the design experience afforded Avery an opportunity to tap into an aspect of division that could be used to reason about even and odd numbers, or what Shulman characterizes in PCK as "aspects of content most germane to its teachability" (Shulman 1986; p. 9).

Casey and Mia also summon such "aspects of content" to inform the design of their tool: they connect numerical considerations about time and its units (such as divisors of 60 and their relationships to fractions) and geometric ones (such as partitioning a circle into purposeful sectors) so that fractional areas can be used to represent features of everyday time telling. If we think of manipulatives as part of curriculum packages, then Casey and Mia were able to take fraction circles, and (re-)interpret and tailor them for the teaching of a special mathematics topic. That is, through the design setting, they were able to modify an existing manipulative and support their learner in reasoning meaningfully about the topic of time telling. These visions of teaching align with reform-minded initiatives regarding teachers' using curricula in ways that adapt to the particulars of their situations and their students (NCTM, 2000; Zumwalt, 1989).

The latter observation segues the discussion into our second question, *How does examining and describing this (design) process help us distinguish a "reform-minded" development and demonstration of PCK?* The three PMTs' intentions to counter children's rote and mechanical approaches to their respective mathematics topics reflect reform-minded calls that recognize how memorization without understanding generates learning that is surface-level and fragile (NCTM; 2000, 2014). Instead, the PMTs were able to reorganize the limited understandings of students who see their topics through traditional techniques, into mathematical learning for understanding that is deeper and more robust. That is, the design experience afforded a newfound PCK of possible "strategies most likely to be fruitful in reorganizing the understanding of learners" (Shulman, 1986; p. 9) for the PMTs.

For both working groups, the tools' designs are intended as a means for children to *reason* through posed tasks, another goal of reform-minded teaching. That is, the PMTs' focus on encouraging a child's reasoning and explanations about a mathematical concept is embedded into the design of each tool (Pratt & Noss, 2010) by way of the learning opportunities each design affords. One aspect of mathematical reasoning that surfaces in both working groups' experiences is the unforeseen and unanticipated occurrences that can arise in genuinely, open-ended situations (see Borasi, 1994; Middleton, Tallman, Hatfield, & Davis, 2015). In classrooms, such incidents can challenge teachers to *revise* or *modify* their thinking or actions in the moment (Borasi, 1994; Confrey, 1990; Shulman, 1986). Because a design experience is, by its very nature, "a process that is spontaneous, unpredictable, messy, creative, and hard to define" (Koehler, Mishra, Hershey, & Peruski, 2004; p. 32), reasoning through it will entail a continual negotiation and renegotiation of knowledge. In the current study's experiences, Avery was challenged to modify the objects for her tool's compartments and additionally reflected on struggles to understand the thinking of a child's justifications for a posed problem. Casey and Mia used their tool's design to write a task that ends up puzzling their child, but also to help him persist and reason through his initial learning dilemma. By prioritizing reasoning, Avery, Casey and Mia are afforded worthwhile opportunities for the coupling of action and reflection that is so pivotal for teacher development, and for experiencing some of the complexity and uncertainty in teaching that often cripples preservice teachers' abilities to engage in classroom investigations (Ghousseini, 2009).

Conclusion

Teacher education is currently being challenged to experiment with approaches that engage teacher educators "more closely with schools in a mutual transformation agenda, with all the struggle and messiness that implies" (Darling-Hammond, 2006; p. 302). Our study presents another avenue for teacher education to serve as a transitioning vehicle toward some of "the struggle and messiness" that occupy schools. Through its inventing and sharing of learning manipulatives, the Making experience provided the PMTs an opportunity to engage in the messy classroom realities of open-endedness and tangents that lessons can give rise to. In providing the three PMTs with the open-ended opportunity to challenge, explore, and extend their mathematical thinking, the design experience elicited authentic teacher learning moments by inviting and supporting continual renegotiations of the special relationship between pedagogy and content, thereby impacting PCK.

We remain hopeful that our project and other constructionist-centered activities can contribute to the avenue of research that conveys the possibilities for PMTs' learning and knowledge development within teacher education. Through our TMML project, we will continue along this trajectory to further characterize, describe, and understand PMTs' knowledge for teaching, thereby promoting the reform-minded ideals of teaching, learning, and doing mathematics within teacher education.

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