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Vignettes of Research on the Promise of Mathematical Making in Teacher Preparation

In this chapter, we share research that explores the potential benefits of a novel Making experience within mathematics teacher preparation that we hypothesized would inform the pedagogical and curricular thinking of prospective teachers of elementary mathematics (PMTs). That experience tasks PMTs with digitally designing, 3D printing, and sharing an original manipulative with a child to support and promote their mathematical reasoning and understanding. With a focus on the design of new tools to support teaching and learning through the use of learner-centered design practices and digital fabrication technologies, this experience has prospective teachers exploring at the intersection of content, pedagogy, and design. We begin by sharing the findings of a pilot study that revealed a surprising breadth of teacher knowledge leveraged by PMTs through their Making activity. Those findings convinced us of the promise of a Making experience within mathematics teacher preparation. They also convinced us to pursue a research trajectory aimed at discerning what other benefits the experience might offer. We share several vignettes of research on that trajectory that take a variety of theoretical and methodological approaches to address research questions at the intersections of teacher identity, teacher knowledge, pedagogy, and design. We provide implications of our findings for teacher preparation and professional learning throughout the chapter and its conclusion.

1 Introduction

Prospective elementary teachers (PMTs) have been characterized as coming to teacher preparation with limited conceptions of mathematics (AMTE, 2013) and a model of mathematics teaching that is oriented more toward the transmission of rules and procedures (Ball, 1990; Ma, 1999; Thompson, 1984) than to the cultivation of conceptual understanding. Consequently, teacher preparation must offer opportunities that challenge this model of mathematics teaching and learning, and provide gateways to meaningful interactions and deepened understanding of both content and pedagogy. Connecting with a body of research that conceives of Making in education as the creative practice of designing, building, and innovating with analog and digital tools and materials (Halverson & Sheridan, 2014), we present one such opportunity that we centered in a novel Making-oriented experience within mathematics teacher preparation. That experience tasks prospective teachers of elementary mathematics (PMTs) with digitally designing, 3D printing, and sharing an original manipulative with a child to support and promote their mathematical understanding. In seeking to determine what this experience might offer PMTs as they prepare for the work of mathematics teaching, our work has pursued a number of theoretical directions and methodological approaches to address research questions at the intersections of teacher identity, teacher knowledge, pedagogy, and design.

Schad and Jones's (2019) review of the research on the Maker movement in K12 education finds that students' learning through Making dominates that literature, with foci that include the improvement of STEM learning outcomes, increasing student motivation and interest in STEM, and increasing equity by broadening notions of what counts as Making in STEM education. The extent to which Schad and Jones's review mentions research on what *teachers* learn through Making is through studies of how they learn to design and run makerspaces, and how they learn to integrate maker-centered learning strategies (Clapp et al., 2016) into their own curriculum. Thus, there is almost no research on supporting teacher learning through Making. We situate our work (Authors, 2017, 2018, 2019, 2020, 2021, under review) within that gap in the research.

In this chapter, we share vignettes of several research projects that address our larger project's broader research question: *What are the potential benefits of a Making experience within mathematics teacher preparation?* These vignettes provide snapshots of our work; they also direct the reader to where they can read more about it. As the Making experience we designed is central to each of these vignettes, we begin by describing it in detail in order to enable a grounding for the theoretical and practical rationales that are presented afterwards. Then, we present the pilot project that eventually became the launching point on the research trajectory of our other projects. Next, we organize the presentation of our other projects into two sections. In Section 2, we present those that occurred within the design environment of PMTs' making. In Section 3, we present those that occurred outside of it and within approximations of practice (Grossman et al., 2009).

1.1 Making in Mathematics Teacher Preparation

This work connects with a body of literature that frames *teachers as designers* (e.g., Brown, 2009; Maher, 1987) of learning experiences and of the material resources that mediate them. We conceive of design quite broadly to include the "intentional activity of transforming ideas and knowledge" (Carvalho et al., 2019, p. 79) into "tangible, meaningful artifacts" (Koehler & Mishra, 2005, p. 135). Our purpose in doing so is to introduce a pedagogically genuine, open-ended, and iterative design experience into mathematics teacher preparation that is centered on the Making of an original physical manipulative for mathematics teaching and learning. We hypothesized that the experience would afford unique pathways of diversified engagement that could promote an epistemic shift toward inquiry-oriented creative and participatory practices that support teaching and learning mathematics with joy and understanding. Accordingly, we view this Making experience from a constructionist perspective (Harel & Papert, 1991), which argues that meaningful learning happens through the designing and sharing of digital or physical artifacts “that learners care about and have some degree of agency over” (Schad & Jones, 2019, p. 2). Indeed, when teachers take agency over the design of their own curriculum materials, they assume ownership over them and the learning environments they generate and come to see themselves as agents of curricular and pedagogical reform (Leander & Osborne, 2008; Priestley et al., 2012).

1.2 Curriculum Context

Data collection for this research took place over three implementations of the Making experience between the spring of 2017 and the spring of 2020. The study occurred within a specialized mathematics content course for PMTs at a mid-sized public university in the northeastern United States. Although the university is a Hispanic Serving Institution (HSI), the majority of the students in the three classes were not Hispanic. Over 90% of all three classes of participants identify as female. These demographics are typical of the prospective elementary teacher population.

Situated in an instructional context in which the teacher educators of the courses practiced an inquiry-oriented pedagogy grounded in a constructivist theory of learning, the course engaged students in a Making experience defined by the following task: “The purpose of this project is for you to 3D design and print an original physical tool (or ‘manipulative’) that can be used in teaching a mathematical idea, along with corresponding tasks to be completed by a learner using the tool.” To realize their project, the PMTs learned to use the web-based Tinkercad (Autodesk, Inc., 2020; see Figure 1, left) digital modeling platform. The design of their manipulative and a corresponding set of problem-solving tasks aimed to reflect a) PMTs’ knowledge of what it means to do mathematics and how we learn with physical tools, b) their knowledge of elementary-level mathematics content, and c) their perspective on pedagogy and curriculum in mathematics education. In addition to the design of the tool, four written project components comprised the data corpus: 1) a “Math Autobiography” that calls on students to reflect on their experiences as a student of mathematics and consider how those experiences might inform their future work as mathematics teachers; 2) an “Idea Assignment” that describes PMTs’ initial thoughts about a manipulative they want to create; 3) a “Project Rationale,” which is an account of how their design reflects an understanding of what it means to know and learn mathematics; and 4) a “Final Paper/Reflection” that presents findings from a “Getting to Know You” interview and problem-solving interviews conducted by the PMTs with their tool and an elementary-age target student.

Although the particular approaches varied somewhat across the three implementations of the Making experience, in each instance, PMTs worked on their designs during in-class design sessions during three or four of the thirteen weekly class meetings. All design sessions – as well as all other class meetings – were held in a large design lab (Figure 1, right), which we chose deliberately because we imagined that the PMTs' design activity would be more inspired in an environment intentionally configured with affordances to support it and that can accommodate the kind of immersive, collaborative social space that nourishes it. Digital fabrication technologies, including about forty 3D printers and two laser cutters, lined the perimeter of the space, as did bookcases of evocative 3D-printed objects designed by students in other courses.

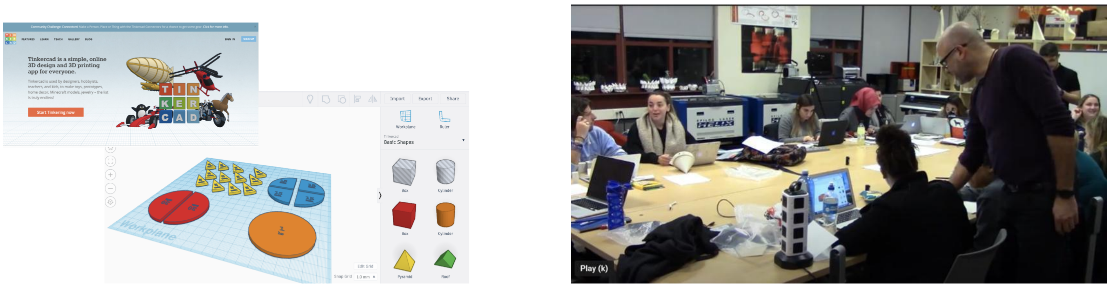


Figure 1. The Tinkercad design environment (left) and the design setting (right).

1.2 Practical and Theoretical Rationales

As we have already alluded to in passing, the design of the Making experience is grounded in the learning theories of constructivism and constructionism. These theories recognize that knowledge is actively constructed by a learner, with constructionism adding the dimension that the knowledge should be constructed during the process of making a shareable object (Harel & Papert, 1991). Then we took a *Learning by Design* approach (Koehler & Mishra, 2005) to leveraging and potentially advancing this knowledge. Learning by design engages PMTs in the active inquiry, research, and *design* – or the purposeful imagining, planning, and intending – that precedes and interacts with Making. In doing so, it honors the proposition that it is productive to develop teacher knowledge within a context that recognizes the interactions and connections among these constituent domains of knowledge. The approach has methodological advantages, as well, since it opens a window into the interplay between a PMT's iteratively evolving artifact and the application of teacher knowledge domains in the artifact's development. Indeed, Pratt and Noss's case study (2010) offers a proof of concept that a learning by design approach provides a venue for characterizing the interplay among a participant's knowledge domains as they are invoked during the design process.

1.3 The Pilot Study

The pilot study for this project was an exploratory one. It took place in the spring of 2017 and its intention was to broadly discern the value of engaging PMTs in Making and design practices that were made possible by increased access to human-centered design practices and digital fabrication technologies, and that we hypothesized would inform their pedagogical and conceptual thinking. Our analysis of the data had its genesis in a more narrow focus, which is expressed in the following research question: *What forms of knowledge can be brought to bear on prospective elementary teachers' design work as they Make new manipulatives to support the teaching and learning of mathematics?*

1.3.1 A teacher knowledge analysis of pedagogical content knowledge in interaction with design activity

We took a grounded theory approach (Corbin & Strauss, 2008) to the analysis of PMTs' written project artifacts using the teacher knowledge literature (e.g., Ball et al., 2008; Koehler & Mishra, 2009; Shulman, 1986) to establish base codes followed by iterative analyses of design cases to generate new ones. We found from that analysis that students used a variety of forms of knowledge in the course of their design activity, including knowledge of mathematics, specialized mathematical knowledge, knowledge of standards and curriculum, knowledge of research on student learning, and knowledge of how students learn with tools as informed by a constructivist perspective. PMTs wrote about how manipulatives aim to embed mathematical knowledge, how ideas may be abstracted to construct ideas through students' manipulations of their tools, and how learners can use their manipulatives as tools to learn through problem solving rather than memorization. They also spoke to how affective concerns (e.g., how their tool might be fun and engaging) and personal experiences as (e.g., struggling) informed their design ideas. A complete articulation of these results that provides evidence of each identified form of knowledge can be found in (Authors, 2017). We only wish to mention that the surprising range of knowledge that PMTs brought to their design activity convinced us of the promise of a Making-oriented experience within mathematics teacher preparation. Consequently, we were convinced of the promise of further exploring the potential benefits of that experience.

As a next step, we took a revelatory case study approach (Yin, 2009) to uncover deeper insights that could help us better understand what we deemed to be an underexplored phenomenon. Specifically, we sought to discern a relationship in the emergent interactions between PMTs' pedagogical content knowledge (PCK) and their design activity. Working individually and asynchronously, two of the authors took their own grounded theory approach (Corbin & Strauss, 2008) to open coding an initial pass of the data using base codes derived from the literature on teacher knowledge. And in an iterative fashion, following each independent review of the data, they came together for a collaborative consultation where they shared and refined their codes and interpretations (Patton, 2015). At the culmination of that analysis, they had each selected the same two working groups for case studies: "Avery" (working alone) and "Casey and Mia" (working together). The researchers shared that they were each drawn to the ways these PMTs made authentic connections between mathematics and the tools they were designing. Here we share some of the findings of the case of Casey and Mia's "Minute Minis" project (see Figure 2). That case is presented more thoroughly in Authors (2019, under review) along with the case of Avery's "Even Number Tool."

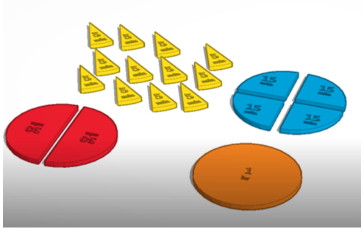


Figure 2. Casey and Mia's Minute Minis

1.3.2 Findings

Casey and Mia were inspired to design a manipulative that could help children reason about the abstract concept of time. In their design rationale, they hypothesized about breaking through the ordinarily obscure nature of time to make it more accessible to learners: "The main goal of this project is to give a concrete representation of the relationship between hours and minutes. Using manipulatives is especially important when exploring new concepts, and [since] time is a very abstract concept, it is especially pertinent that students have something concrete to work with." This idea came from Casey, whose thinking was informed by coincidental work as a student teacher:

> Currently, most of the 2nd graders in my class can tell time to the nearest half hour, yet I am unsure of how they know how to do this. Is it just because they know that when the minute hand is pointing at the 6, I say _:30 and when it's pointing at the 12, I say _:00? Or do they have a… deeper understanding of time and how a clock works?

These considerations reflect how Casey's PCK (wondering about students' conceptions of time) informed her design. Over the course of the project, these questions developed into other strands of knowledge that she and Mia used to investigate these issues as the two were driven by a desire to transform potentially limited conceptions of time from memorized models into deeper mathematical meanings. Drawing upon other aspects of PCK and of mathematical and curricular knowledge, Casey and Mia took an existing design of fraction circles and used concepts from geometry to amend it for their objectives. They wrote:

> [We] will be using the same concept of fraction circles, yet instead of labeling them with a fraction, they will be labeled with minutes. For instance, a whole circle will be labeled '1 hour,' while two half circles will be labeled '30 minutes.' [We] will also have [fraction] circles for 15 and 5 minutes.

One of Casey and Mia's key design decisions focused on being able to "visually illustrate the concept of minutes as fractions of an hour." The circular shape was important to them in ensuring "that students would be able to use the Minute Minis directly on the face of a clock. This would aid [them] in exploring the relationship between where the minute hand is pointing and the number of minutes past the hour." They deemed this design affordance essential in supporting student inquiry of the fractional ideas embedded in the tool so that the child could assemble the fractional pieces to compute time.

In summary, Casey and Mia shared reflections that leveraged their knowledge of fractions and area to mediate a bridge between abstract and concrete representations of time. By supplementing the traditional focus of instruction about time with a concrete representation that facilitates conceptual connections between a clock face and its underlying area properties, they drew on this knowledge to articulate the mathematical richness underlying their manipulative and its possible uses by a child.

1.3.3 Implications

As PMTs assumed the multi-faceted role of teachers as designers of instruction within a space of technological possibilities, they created powerful and innovative tools, and their work demonstrated a rich and mature repertoire of knowledge domains that we are not typically afforded opportunities to see (AMTE, 2013). The identification and advancement of this knowledge suggested to us the promise of a Making experience within mathematics teacher preparation and convinced us of the value of discerning what other benefits the experience might offer. Accordingly, our pilot work became the launching point on that trajectory of research.

2 Knowledge Interacting with Design

With the theoretical and practical rationales for the Making experience now laid out, and with evidence from pilot work that speaks to the potential benefits of that experience, we now present vignettes of other research we have conducted on this trajectory. In this section, we share findings of studies that occurred within the design environment of PMTs' Making. Then, in Section 3, we present those that occurred outside of it and within approximations of practice. We made this distinction in our research as we sought to explore the potential for transfer of PMTs' learning from teacher preparation and into their practice. We did so, because that connection too often proves rather difficult to sustain.

2.1 The Interwoven Discourses Associated with Learning to Teach Mathematics in a Maker Context

Recent conceptualizations of teacher knowledge build on previous characterizations of distinctive knowledge domains in order to promote a wider focus on their integration (Scheiner et al., 2019). In this phase of our project, we adopted this perspective by viewing *teachers as learners* and foregrounding their identities (Sfard & Prusak, 2005) in order to recognize what affective, interpersonal, and social matters can bring to this conversation. That is, by honoring the interrelationship between the learning of mathematics and the learners themselves, the promise of this approach is suggested by the proposition that teachers' "invention[s] of 'objects-to-think-with'... [offer] the possibility for personal identification" (Papert, 1980, p. 11).

2.1.1 A discourse analysis of identity in interaction with mathematical and pedagogical design activity

We adopted a commognitive perspective on learning (Sfard, 2007, 2008), which is one that encompasses both interpersonal communication and individual cognition. Our objective in doing so was to explore the premise that learning to teach mathematics can be seen as changes in discursive activities that include narratives about mathematics and identity (Heyd-Metzuyanim & Sfard, 2012; Sfard & Prusak, 2015). The following question guided the research: *As prospective teachers of elementary mathematics Make new manipulatives to support the teaching and learning of mathematics, what might their discourses reveal about the epistemology of learning to teach mathematics?* We addressed the question through a revelatory case study (Yin, 2009) of a prospective elementary teacher named "Moira" and by framing mathematics learning as the interplay between discourses about mathematical objects (*mathematizing*), participants of the discourse (*identifying*), teaching and learning (*pedagogy*), and design activity (*designing*). This framework provided us with a lens through which to study how the process of making a manipulative can provoke the four discourse activities and make visible the intertwined nature of a teacher's learning. We chose Moira because her initial design was a tool intended to simulate the "keep-change-flip" algorithm for fraction division. However, when the course's teacher educator pushed back on the idea because it did not meet the project's expectations for a tool that would support a students' conceptual learning, she tried to make sense of the algorithm but could not, and eventually she abandoned the idea altogether. We sought to understand this change through the lenses of the four discourses.

2.1.2 Findings

In this section, we present just two central results. These came from a follow-up interview we conducted with Moira in order to understand her rationale for the change in her design idea. The first concerns our analysis of this change through the discourses of Mathematizing [M], Pedagogy [P], Designing [D], and Identifying [I], and is as follows:

> "I wanted to make something that could be interpreted in many different ways" [M/P/D], she shared, "that wasn't something that I was just forcing them to, like, all right, you have to use it this way. I wanted it to be able to be manipulated" [M/P/D/I]. As she considered her initial "keep-change-flip" tool, she explained how she realized that, "flipping the fraction upside down in my initial tool... it was just not useful [M/P/D] ... So I decided to switch to comparing fractions and then I came up with this [fraction comparison tool]" [M/D/I].

These reflections revealed how Moira's decision to abandon her fraction division design was not just about mathematizing, it was also about identifying. As a teacher, it was important to her that her students have the opportunity to develop their own ways of thinking about fractions with a tool that can be used in a variety of ways. Moira acknowledged that the pedagogy promoted by the instructor in the classroom was also part of her decision to change her design: “Well, [the change of design] was because we were talking and you [the teacher educator] said, ‘you’re just teaching them how to – you’re just giving them a way to solve the problem.’ And I realized, you’re right ... It wasn’t helping them learn how to do a problem” [*M/P/D/I*]. By switching to a design for comparing fractions, Moira found that she could participate in the discourse endorsed in the course and honor the teacher she wanted to be.

A second result emerged from our awareness that her current tool (see Figure 4, left) *could* be used to make sense of fraction division and a question about whether Moira realized this capability in her tool. Our query to her about this possibility using the problem, ½ divided by 2, prompted Moira’s in-the-moment reflections such as this one: “½ divided by 2. ½, this divides it into two equal parts, and I know this equals fourths, so this is ¼” [*M/I*]. Next, as she was investigating 1 divided by 1/3, Moira took the 1 and 1/3 ring, guessed that the answer was 3, and said, “I know I can do it, and I’m seeing it, but I don’t know how to describe it” [*M/I*]. Moira was using her tool to make sense of this problem when we prompted her to explain whole-number division (e.g., 6 divided by 3). As she reasoned through whole-number examples [*M*], she exclaimed, “Oh! So, so, if I am dividing 1 by ⅓, there are three thirds in 1, so it’s 3! Yes! You can do division with these… Wow! Fractions make so much sense now” [*M/I*].

Moira’s shift from the use of a partitive conception of division to a measurement one gave her sought-after language to describe her tool’s utility in her understanding of fraction division. That mathematical discovery was intertwined with an expression that revealed how emotionally invested she was in this realization. As she used her tool to think through fraction ideas [*M*], Moira came to recognize its potential not only for her own learning, but also for teaching fraction division in a way that aligned with her identity as a teacher [*P/I*]. Her expressive body language and energy substantiated her enthusiasm for the discovery.

2.1.3 Implications

As in a woven tapestry, learning to teach mathematics weaves together four threads – or discourses – that are unique to a PMT’s discursive experiences and particular to a learning community where an inquiry pedagogy is promoted. In this sense, to characterize Moira’s learning to teach mathematics as a complex structure of discursive activities interwoven in dialectical unity is to illuminate the brilliance of a tapestry threaded by what she wants to teach (mathematizing), how she wants to teach it (pedagogy), decisions about what resources to make available (designing), and the kind of teacher she wants to be (identifying). Collectively, these threads contribute to an apparently honest depiction of the “organic whole” (Scheiner et al., 2019, p. 165) that is learning to teach mathematics.

This finding of the intertwined nature of the four discursive activities establishes that identity is as central to learning to teach mathematics as is the learning of mathematics, pedagogy, and design. And its implications speak to the potential of interdisciplinary experiences like the design experience as venues for the meaningful learning of learning to teach mathematics within teacher preparation coursework.

2.2 Making as a Window Into the Process of Becoming a Teacher: The Case of Moira

The process of becoming a mathematics teacher entails the development of an integrated knowledge base including both content and pedagogy. Much research has been done to better understand the forms that this knowledge base might take and how its development in PMTs might be supported. What is less clear is an image of the experiences these PMTs find formative as they progress through their teacher preparation coursework. Thus, in this phase of our project, we sought to illuminate the processes of *a teacher becoming* as they are mediated by a variety of social and conceptual resources within teacher preparation. The following question framed the inquiry: *How does a Making experience in mathematics teacher preparation mediate the social and conceptual dimensions of the process of becoming a teacher?*

We addressed this question through a revelatory case study (Yin, 2009) in order to better understand the processes at play in design activities that were hypothesized to underpin moments on a trajectory of becoming a teacher. Again we chose Moira, and we did so because, more than any other student, she *designed aloud*. Her words, gestures, and other embodied actions gave unprecedented access to elaborate and interwoven discourses of mathematics, pedagogy, identity, and design in Moira's design conversation (Schön, 1992). We were able to document her trajectory of becoming via an analysis of those discourses.

2.2.1 Framing making as mediated learning

We brought situated and sociocultural theories of learning to bear on our attention to practice and to the social and conceptual artifacts that mediate knowledge and identity formation through that practice. In particular, we grounded this work in Engeström's (1987) cultural-historical activity theory (CHAT) and Holland et al.'s (1998) concept of figured worlds. The CHAT perspective offered a lens with which to situate Moira's activity in relation to the design context in which it arose. At the same time, a figured worlds perspective captured the mediating role of social, material, and conceptual artifacts on learning as identity formation. Together, the two perspectives proved useful in our analysis of the data to reveal and craft narratives of salient moments in Moira's becoming a teacher.

2.2.2 Findings

At the conclusion of our analysis, we chose four "moments of becoming" from the collection of moments that had been coded in our analysis. Their titles appear below along with brief descriptions. The entirety of these narratives appears in (Authors, under review). We describe them briefly here.

2.2.2.1 Moment 1: *"I should not be allowed in this class. I'm having too much fun."*

This moment explains how an assemblage of design-related practices offered a space of authoring (Holland et al., 1998) for Moira's identity. Iterative enactments of agency involving conceptual, material, and social artifacts mediated her identity development as she "reconceptualize[d] what and who" (Vågan, 2011, p. 49) she was from one lived moment to the next. In addition, within that space of authoring, Moira was afforded a space of "improvisational play… [the] predominant form of agency" (Chang, 2014, p. 33) – the "medium of mastery, indeed of creation, of ourselves as human actors" (Holland et al., 1998, p. 236). Moira's improvisational play appeared to have been fueled by creative tendencies that she described as being embraced in elementary mathematics but later suppressed through the message that "math and science will probably come difficult" to students with "creative minds." In contrast, these creative capacities were not only invited into the design session, they were "demand[ed]" in response to the "possibilities of a design situation" (Schön, 1992, p. 4). Furthermore, as she was sought out by classmates for help with their designs, she was positioned by them as an expert. The community-subject interactions that mediated the rules by which participants related to each other spoke to the situated nature of activity in the figured world that emerged for Moira as she was once again positioned as a "top student," just as she was in elementary school.

2.2.2.2 Moment 2: *"Ugh, you're right. We can't flip the first one. I'm gonna work on this."*

As we analyzed the evolution of Moira's design within this Moment, we came to realize the saliency of several artifacts that mediated her design decisions. A lived history of all-too-common experiences in the figured worlds of traditional mathematics classrooms informed her interactions with fraction concepts and the concept images (Tall & Vinner, 1981) that were co-determined in them. In addition, pedagogical commitments and her knowledge of content and students (Ball et al., 2008) interacted with design activity that was further shaped by broader flow of social interactions distributed across the design environment. Taken together, we were reminded of Roth's (2012) assertion that we are only able to understand the actions of a subject on the object of their activity when we consider all the relations that mediate every aspect of the activity. Thus, this situated network of individual and collective activities that mediated the distinctive evolution of Moira's manipulative offered concrete markers of the formation of her identity as she emerged as a more central participant in an educational design community.

2.2.2.3 Moment 3: *"Do I know what a torus is? No. Am I using it? Yep!"*

In this third moment we centered our analysis on Moira's mathematics and how it mediated her design activity. From a mathematical perspective, equal partitions of a unit whole express the part-whole relationship that is fundamental to fraction knowledge. From a design perspective, the notches that form these partitions embed that part-whole relationship into Moira's manipulative (see Figure 4, left). In addition, aesthetic considerations regarding the form of those notches mediated Moira's design as she decided to cut them using a torus, a mathematical object she had hardly been familiar with but whose affordances she discovered in action. In this regard, we found evidence of the "double arrow" (Engeström, 1987) of mediating interactions between mathematics and design.

2.2.2.4 Moment 4: *"I just figured out so much math!"*

In this moment of becoming, as Moira realizes that she's used her tool to *learn* fraction division, she's also realized her initial objective, the one she had abandoned and the one she had now achieved, which was to design a tool to *teach* fraction division. And yet again, affect and cognition arose as manifestations of the same accomplishment (Roth, 2012). In an act that is a manifestation of community relations (Engeström, 1987) that had the researchers participating alongside Moira in her quest to make sense of fraction division, everyone celebrated along with her. "Wow," she exclaimed. "Fractions make so much sense now. That blew my mind... I've learned so much. This is a great day!"

2.2.3 Implications

The use of CHAT and figured worlds perspectives made apparent the formative power of the Making experience by affording Moira opportunities to make choices that leveraged her mathematical, pedagogical, and design knowledge and moved her along a trajectory of a teacher becoming. This finding contributes to the research on teacher learning and identity formation in teacher preparation. In addition, by further demonstrating the value of accounting for identity formation in research on mathematics teacher education, this finding also generates new opportunities to move the field forward in relation to research into the potential value of constructionist, STEAM-integrated curricular experiences in teacher preparation.

2.3 The Nature of Prospective Mathematics Teachers' Design Activity as they Make Original Manipulatives

Positioning teachers as designers of their own curricular resources invites opportunities for their explorations of innovation at the intersection of content, pedagogy, and design. And given that there is almost no research on supporting teacher learning through Making outside of our own project, this gap in the research misses the opportunity to explore the design decisions teachers make through their design activity. This vignette addresses this gap as it poses the following question: *As prospective mathematics teachers Make new manipulatives* *for mathematics teaching and learning, what is the nature of the resources and rationales they bring to their design decisions and how do these intersect to mediate their decision making?* We address this question through an exploratory case study approach (Yin, 2009) to understand PMTs' design activity by considering the three elements of each of their design decisions (Figure 3). Then, we share findings from the analysis of this activity that convey the diversity of design decisions, rationales, and mediating resources that it entailed.

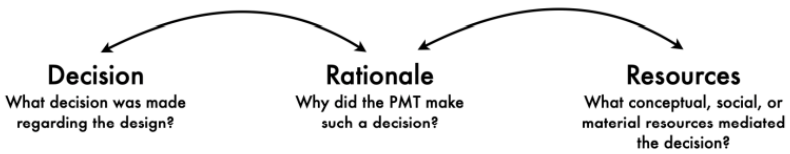


Figure 3. The 3 elements of a design decision

2.3.1 Theoretical Framing

Grounding this work in a Learning by Design approach (Koehler & Mishra, 2005) enabled us to characterize the interplay between a designer's knowledge, experiences, intentions, and other resources as they are invoked during the iterative design of the shareable object. In addition, Schön's (1992) notion of "knowing in action" (p. 2) enabled us to characterize and organize the resources that mediated PMTs' design decisions. That notion provides that a "designer sees what is 'there'…, draws in relation to it, and sees what he/she has drawn, thereby informing further designing" (p. 5).

2.3.2 Findings

Here, we present the cases of "Moira" and "Anyango," as their written work expressed the greatest number of design decisions from among the thirty-four projects we analyzed. In the subsections that follow, we share and contrast three of their design decisions.

2.3.2.1 Student-centered design

Both Moira and Anyango chose the child they had worked with in problem-solving interviews earlier in the course as their intended user of a tool. Moira explained that the student she tutors was having trouble with fractions, so she decided to create a tool to help him. Anyango shared that the student she was working with said she enjoys fractions, so she wanted to nurture that interest. Although both Moira and Anyango decided to create a fraction tool, they provided different rationales for that decision. Moira hoped to help her child make better sense of fractions; Anyango hoped to extend her child's current thinking about them.

2.3.2.2 The nature of the tools

Moira's design (Figure 4, left) is a tool for fraction comparison. It consists of "a series of rings that rest on a cylinder... The notches help divide the rings equally up into pieces to represent parts of a whole. Each ring represents a different number of parts, like sixths and eighths." Anyango also designed a tool for fraction comparison (Figure 4, right), yet her design is markedly different from Moira's. Anyango described her tool as "a 3D version of fraction strips. Each strip was made to be a rectangular/square piece that slides into individual pegs…[the] blocks stack vertically... to indicate height as value and amount."

Figure 4. Moira's (left) and Anyango's (right) fraction tools.

While the mathematics of fractions and the knowledge of technology mediated both their design decisions, fraction concepts are embedded differently in their designs. For Moira, these are represented as arc lengths of the partitions of a continuous ring; for Anyango, these are represented as discrete fractions of the height of a referent whole. Relatedly, Moira's imagined utilization scheme involves aligning notches so that, for example, "the rings are able to be compared, showing how many fifths are in one half." Central to Anyango's scheme is that "all the fractions [can be] mounted on one platform… so that the student could begin to grasp how all the smaller parts can equate and compare to the whole."

2.3.2.3 The role of aesthetics

To Anyango, "The colors didn't matter much." Giving each fraction block its own color would have been "aesthetically pleasing, but it did not affect how the manipulative worked." Moira made the same design decision, but with a different rationale mediated by different conceptual resources. She explained that all her "rings have the same color," because if each ring had a unique color, it might "take away reasoning from children. If a student believes that a yellow ring represents 1/6ths, they will immediately reach for yellow the second that they hear sixths." By giving the rings the same color and leaving them "unmarked," Moira ensured that children will construct their own meanings in relation to each of the rings, thereby giving her tool the promise that it can "be used in multiple ways." Thus, epistemological knowledge mediated a decision that seems to reflect Moira's commitment to an inquiry pedagogy that affords multiple means of engagement.

2.3.3 Implications

As our research continues to discern the intellectual influences of the Making experience on PMTs' pedagogical and curricular thinking, this research adds more nuance to findings from prior research that revealed the breadth of teacher knowledge that they bring to their designs. It does so by revealing the particular power of design activity – namely the diversity of design decisions that PMTs must make in order to meet the expectations of the experience – to elicit, articulate, and advance that knowledge. Thus, this finding speaks to the generative power of an open-ended and iterative design experience in terms of the agency prospective teachers enact throughout their design activity and the wealth of knowledge and experiences that mediate it.

3 Knowledge in Practice

In the previous section, we shared vignettes of research on activities that took place within the design environment of PMTs' making. In order to explore the potential for transfer of PMTs' learning from the design setting and into their practice, in this next section we share vignettes of research that occurred within approximations of practice. We begin with a vignette that extends the one just presented in Section 2.3, which identified the knowledge resources PMTs brought to bear on some of their design decisions.

3.1 Prospective Mathematics Teachers' Designed Manipulatives As Anchors For Their Pedagogical and Conceptual Knowledge

This next vignette presents research that sought to discern whether connections could be made between the pedagogical/conceptual knowledge that PMTs construct in teacher preparation and how that knowledge is enacted in their teaching. We wondered whether their designs could possibly mediate – or be some sort of anchor for – their pedagogical visions. Specifically, we asked the question: *As prospective teachers Make new manipulatives for mathematics teaching and learning, can connections be made between the pedagogical/conceptual resources for their design decisions and how those designs mediate the pedagogical moves they make in practice?*

3.1.1 Research Design

To answer this question, we took a sociocultural perspective and grounded this work in the notion of mediated activity, derived from Vygotsky (1978) and advanced as instrumented activity by Verillon and Rabardel (1995). We use the term *embedding* to connote an intentional (Malafouris, 2013) design element that embeds a PMT's pedagogical and/or conceptual knowledge in their tool. PMTs do so, so that in practice, the tool can afford (Gibson, 1977) particular utilization schemes (Verillon & Rabardel, 1995) that the PMTs hypothesized would enable the child to abstract (Piaget, 1970) percepts that are the constitutive elements of concepts. We then took an exploratory case study approach (Yin, 2009) using grounded theory (Corbin & Strauss, 2008) to discern instances in PMTs' teaching when the use of their artifact implicated the pedagogical and/or conceptual knowledge underlying their design rationales. The locus of these particular research efforts among the broader research project is depicted as the arrow from "Design Decision" to "Enactment" in

Figure 5. Note that central to the figure are the elements of a design decision that appear in Figure 3.

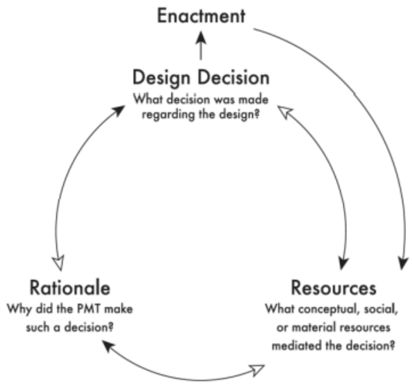


Figure 5. Conceptual resources inform rationales for design decisions and may also be evoked in enactment. Open arrows acknowledge that feedback is reciprocally informing.

3.1.2 Findings

Here we present two excerpts from "Roda" and "Anyango's" tool-based problem-solving interviews that demonstrate how their embeddings of pedagogical and/or conceptual knowledge in their designs served as an anchor for that knowledge in practice. Their manipulatives are shown in Figure 6.

Figure 6. (a) Roda's decimal snake and (b) Anyango's fraction tool.

3.1.2.1 Reasoning about the unit whole

Roda's tool is a "Decimal Snake" (see Figure 6a) that she designed in order to teach a child about decimals and decimal comparison. The tool consists of ten pieces, each equally partitioned into ten parts. Thus, the decimal snake can be used to represent any value between 0.01 and 1 to two decimal places. These design features are Roda's embeddings of the concepts of the whole and its decimal parts. During the interview, when Roda asked her target child to use the snake to compare 5.5 and 5.47, the child manipulated the tool and used it to demonstrate his response: "5.47 is 5 and 47 hundredths, because it's 3 hundredths away from 5 and 5 tenths." While Roda's design intention was for the child to *only* compare the decimal parts, after 60 seconds of struggle, the child located 5.5 at (what we would identify as) 0.55 (if the entire snake represented 1), and 5.47 at 0.47. We inferred from his solution that he had *unintentionally* designated each piece of the snake as 1 (as opposed to 0.1) and each partition of a piece as 0.1 (as opposed to 0.01). Thus, he changed his designation of the entire snake from 1 to 10, and consequently, each piece of the snake now represented 1. In such case, 5.5 would be presented as the 5th partition of the 5th piece.

Determined to help the child identify and resolve the confusion, Roda asked the child to "Show me one tenth" on the snake. He pointed to one of the tenth pieces. "Two tenths?" he pointed to the second (tenth) piece. Next, Roda asked, "Where is 5 and 5 tenths?" Roda's questioning perturbed the child's thinking and provoked disequilibrium. As a result, the child declared, "Oh, wait! This [entire snake] is one whole! 5 and 5 tenths, you can't even make it out of the snake!" Roda then leveraged the affordance that each piece of the snake could represent a tenth of a whole to help him resolve his confusion about the representational capacities of the tool.

Roda: You need how many snakes to make 5.5?
Child: You need 5– No, 6 snakes!
Roda: How can we compare [5.5 and 5.47] using 1 snake? Is that possible?
Child: We can pretend that each piece is one snake.

In this excerpt, Roda leveraged her *unintentional* pedagogical embedding of a conceptually resourced design decision that allows for flexibility in naming the unit whole in relation to the snake and its pieces, revealing the student's thinking and posing purposeful questions to advance his mathematical reasoning.

3.1.2.2 Noticing in action

While Anyango's tool (see Figure 6b) is similar to Roda's in purpose, Anyango explained that she designed her tool "to help the student visualize and deepen their understanding as they explore fraction relationships." Her decision to design a tool that affords a vertical stacking of fraction pieces rather than in horizontal arrays allows the child to have "all the fractions mounted on one platform with the 1 (whole) always being visible, so that the student could begin to grasp how all the smaller parts can equate and compare to the whole." Anyango also engraved the fraction name of each piece on one of its lateral faces. In practice, she posed the task, *Jack and his two friends each had the same size pizzas for lunch. Jack ate 5/8 of his pizza. Judy ate 2/3 of her pizza. And Sam ate 3/6 of his pizza. Who ate the most pizza? Who ate the least?* In response, the child stacked five one-eighth pieces, two one-third pieces, and three one-sixth pieces, each on their own pedestal with their labels facing her (Figure 6b, right). Anyango's pedagogical intention was for the child to compare "heights as amount" and identify the tallest as the one "who ate the most,'" and vice versa. In contrast, when asked, "So, if we just look at this, who ate the most?" the child attended exclusively to the symbolic representations engraved on each piece. This led her to decide that, "It's Jack" (represented by the 5/8 piece). She justified her answer by saying that "5 out of 8 is the biggest of all of them… 2 out of 3 is smaller and 3 out of 6 is… kind of small." Then, when Anyango asked the child what made her think it is smaller, the child explained that, "The top is two and the bottom is three." We infer from this response that the child was basing her comparisons on interpretations of fractions not as parts of a whole but as two separate whole numbers. That's why, for the child, 5/8 is greater than 2/3.

We interpret Anyango's next move as a noticing one (Sherin et al., 2011) that leveraged her pedagogical knowledge about the efficacy of attending to, interpreting, and responding to students' thinking:

> *Anyango:* If I turn this [pedestal] around [Figure 6b, left, such that the child's gaze can no longer be restricted to the fraction labels on the pieces], who has the most?
> *Child:* This one <points to the stack of two one-third pieces, which corresponds to Judy's share>.
> *Anyango:* Who has the least?
> *Child:* This one <points to the stack of three sixth-pieces, which corresponds to Sam's share>.

In this excerpt, Anyango's "flipping" move leveraged an unintentional design affordance that we suggest served as an *anchor* for her pedagogical knowing in action mediated by that affordance. Reinterpreting Schön's (1992) concept of "knowing-in-action" as a noticing-in-action, we suggest that in this instance, Anyango saw what was there, made a move in relation to it, and saw what that move accomplishes, thereby informing her next steps. Thus, by returning the tool to its initial, label-facing orientation so that the child could connect the physical representation of the amount to the symbolic one, the child was supported in determining a correct response to the question, "Who ate the most?"

3.1.3 Implications

In an attempt to solve the perennial problem that teachers tend to face considerable challenges in transferring their theoretical knowledge into practice, this work explored teacher learning at the interface between theory and practice by discerning whether connections could be made between the pedagogical and conceptual knowledge that PMTs construct in teacher preparation and how that knowledge is enacted in their teaching. As PMTs used the manipulative they designed in a problem-solving setting, we analyzed instances when their manipulative served as a mediating anchor for the pedagogical and conceptual knowledge they acquired in teacher preparation and subsequently embedded in their designs. In relation to practice, our identification of these instances of anchoring phenomena suggests that the Making experience yielded material epistemic scaffolding (in physical manipulative form) that supported PMTs and their commitments to the models of knowing and learning they construct in teacher preparation. And in relation to theory, findings from this study and the prior one that it builds upon suggest the analytic value of our design, rationale, resource, and practice (DRR-P) framework for revealing the promise of the Making experience.

3.2 Dare to Care: A Case Study of a Caring Pedagogy on Mathematical Making, Teaching, and Learning

This case study investigates the interaction between a *caring pedagogy* and Making, and how the two informed each other in our project. Because the subject of mathematics (Stinson, 2004; Gutiérrez, 2017) and the Maker culture (Barton et al., 2017) can be interpreted as exclusionary to so many students, we wondered about the possibilities that *caring pedagogies* could bring in broadening opportunities for learning and learners in these spaces. The three central participants in this section include the teacher educator (TE) and the PMT, "David," each of whom brought caring pedagogies to the project and viewed themselves as interlopers to the Making culture. And then there's "Vincent," a kindergarten student who is on the autism spectrum and whose energetic ways of learning are not typically embraced in traditional mathematical classrooms. By focusing on *caring-centered relationships*, we illustrate how *together*, the participants redefined values associated with Making, traditional mathematics, and what can get celebrated as learning.

3.2.1 Framework and Research Question

The fact that Making and caring can elicit both cognitive and affective concerns suggests a need for a framework that accounts for these dual traits. Hackenberg (2010) terms a *mathematical caring relation* (MCR) as one that honors both the mathematical and affective parts of learning. She recognizes a teacher's sensitivity to a student's learning needs and their ability to participate in the activity at hand as central to supporting meaningful MCRs. Hackenberg describes how *cognitive decentering* can help a teacher to navigate an MCR by decentering "from his or her own perspectives… to help students realize and expand their ideas and worlds" (p. 239). In our project, we honor and utilize the mathematically open-ended nature in designing and Making a manipulative; the sometimes, uneasy navigation through emergent mathematical "unknowns"; the child's unique experiences and needs; and the tensions that are negotiated by carers (Noddings, 2012) in balancing these considerations. With this framing, we posed the following research questions: *How does enacting a caring pedagogy during a Making-centered experience impact and broaden opportunities for meaningful mathematics learning? How does this challenge traditional notions of who can Make, who can participate in mathematics, and who cannot?*

3.2.2 Methodology

In exploring the larger question of how the PMTs see themselves as mathematics teachers, we were drawn to *caring relationships* that developed between project participants and utilized the methodological stance of *purposeful sampling* (Creswell, 2007). We opened our analysis to participants' verbal utterances and intonations, body language, actions, and mutual positionings (Simmt, 2000) as revealing defining moments in MCRs. The possibility of intersecting caring theories with Making and the novelty of our data suggested a grounded theory approach (Glaser & Strauss, 1967) to analyzing and cross-referencing our sources.

3.2.3 Findings

Initially, David created a quick and easy answer to the task of Making the project manipulative (by way of designing an already-existing manipulative with a fellow classmate). However, he was invited to reconsider this approach by his TE, who noticed the special and warm interactions between himself and Vincent that David had recorded in a "Getting to Know You" session. Overcoming trepidation of her own, the TE invited David to design a manipulative that responded to these interactions, and made clear that she would support David in this initiative when he realized it would require more time, thought, and *care*. We recognized this as the TE accepting responsibility for supporting David in caring for Vincent and in navigating the discomfort and tensions (Noddings, 2012) that accompany this pedagogical decision. David, in turn, opened to accepting responsibility for Vincent's care by sharing and utilizing Vincent's knowledge and his love of diverse shapes. David attempted to understand Vincent's strengths with shapes, and after a few sessions with Vincent, opted to design triangular, square, and hexagonal prisms with holes and corresponding inserts intended to create a one-to-one matching task (for example, *Which of these shapes fit together?*).

During a design session, David noticed that multiple printed inserts did not fit into their intended holes. The TE took advantage of this moment of struggle to support David through his technological anxieties, and recommended including the extra "mis-shapes" in the matching task (for example, *Which of the multiple hexagonal inserts can fit into the hexagonal hole?*). David reflected on this as being a "teachable moment," such that his "mis-shapes" could become usable for Vincent's learning. In another teachable moment, Vincent showed David how every shape and insert need not match to fill the holes (e.g., Vincent drops hexagonal inserts into the square hole). These uninhibited moments of insight suggested a transition in Vincent's attention from a shape's sides to whether or not it has a hole—a driving force in understanding the concept of topological equivalence. These explorations culminated when Vincent aligned the hexagonal and square prisms with unlike holes to peer through them, and in response, David arranged the pieces between himself and Vincent to form a telescope (see Figure 7)! Together, they locked eyes and exchanged laughter and words of affirmation in an MCR where David decentered from the intended activity to *literally see his child's point of view* (Hackenberg, 2005).

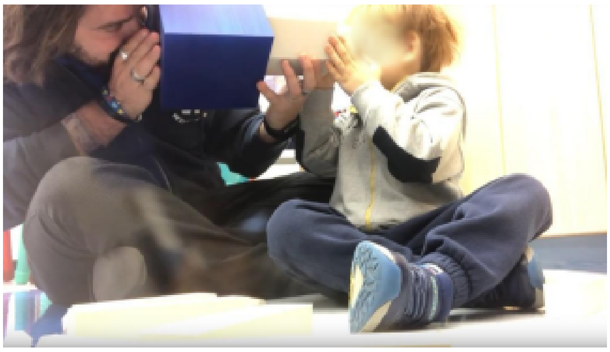


Figure 7. Vincent sees similarities in different-shaped holes.

3.2.4 Implications

The increasing pressures and responsibilities faced by teachers and teacher educators can make enacting caring pedagogies seem especially daunting. Our project's focus on Making something *for* and *with* a specific student enabled both a TE and PMT to leverage their caring-centered pedagogies, and speaks to the inclusivity that caring brings to learning. Vincent, a member of the students with disabilities (SWD) community, approached and demonstrated learning with animated physical enthusiasm. The TE and David enacted caring-centered pedagogies that embraced Vincent's inclination to learn with his body, explore open-ended mathematical ideas *together*, and recognize that design "mis-shapes" could become viable learning tools for Vincent.

By inviting David to substitute a more open-ended investigation for his initial, "easy" project solution, the TE set in motion a ripple effect that challenged traditional notions of mathematics learning in which uncomfortable discoveries such as David's "mistakes" are dismissed as divergent from intended tasks (Lampert, 1990). Instead, David embraced those mistakes as an important part of his learning and celebrated Vincent's mathematical discoveries. In doing so, he defied the exclusionary notion that SWDs should not be expected to participate in problem solving, and welcomed the unexpected (but worthwhile) mathematical interpretations that open-ended investigations can bring. By providing a platform "to demonstrate care for individual students and for the subject matter itself" (Bartell, 2011, p. 54), this case demonstrates how Making and designing can create a novel opportunity to embrace mathematical struggle, surprise, and discovery for all types of learners.

3.3 Harmony and Dissonance: An Enactivist Analysis of The Struggle for Sense Making in Problem Solving

The teaching and learning of mathematics requires teachers and students to give "explicit attention to the development of mathematical connections among ideas, facts, and procedures" (Hiebert & Grouws, 2007, p. 391). Indeed, the National Council of Teachers of

Mathematics (NCTM, 2000) emphasizes the value of representing mathematical ideas in a variety of ways, and that these representations are fundamental to how we understand and apply mathematics. While much research has been done regarding the ways in which teachers can support students' engagement with multiple representations, what is less well understood is the process by which multiple representations of a concept can be leveraged and connected to contribute to learners' meanings of the referent concepts for those representations. Thus, in this phase of our research we aimed to address that gap as we posed the following question: *How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?*. We did so through a revelatory case study (Yin, 2009) of the problem solving of two learners, "Dolly" (a researcher-participant) and "Lyle," as they aimed to make sense of fraction division by coordinating meanings across two artifacts, one being a "Fraction Orange" physical manipulative that Dolly designed and the other being a written expression of the standard algorithm (see Figure 8, left and right, respectively).

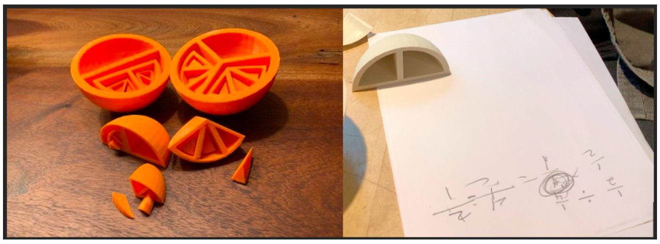


Figure 8. The Fraction Orange and the algorithm

3.3.1 An Enactivist and Semiotic Analysis of Emergent Problem-Solving Activity Involving Multiple Representations of Fraction Division

This study is grounded in the enactivist theory of cognition (Maturana & Varela, 1987), which recognizes that the learning environment (e.g., a task, setting) and the solver(s) (e.g., a student, teacher) are structurally coupled and determined through a dynamic, emergent, contingent, and "ongoing loop" (Proulx, 2013, p. 319) of interactions of problem solving. Verillon and Rabardel (1995) add the dimension that sense making is inextricably linked to the material and symbolic tools that mediate its learning. Thus, we considered what an enactivist analysis might reveal about the processes at play in mathematical meaning making as it develops through the complex interplay of signs and meanings (Maffia & Maracci, 2019) associated with learners' engagement with multiple representations. Maffia and Maracci's (2019) concept of semiotic interference is thus used in tandem with the enactivist analysis to analyze the dynamic, emergent, and contingent (Proulx, 2013) interactions that Dolly and Lyle have with the Fraction Orange and the algorithm.

3.3.2 Findings

While the 13-minute problem solving interview video offers many opportunities worth sharing, here we present just two central moments in order to demonstrate what our theoretical lenses revealed. Note that all fraction pieces of the Orange (Figure 8, left hemisphere) are named in our analysis just as Dolly and Lyle name them: the hemisphere of the Orange is the whole, and that whole is partitioned into halves, fourths, eighths, and sixteenths.

3.3.2.1 Embarking on a path of problem solving

The problem-solving interview opened with Dolly posing the problem, ½ ÷ ¼, on paper alongside her fraction orange. Lyle chose the pen and paper (over the Orange), performed the flip-and-multiply algorithm (½ ÷ ¼ = ½ × 4/1 = 4/2 = 2/1), and declared his answer to be 2. We interpreted Lyle's application of the standard algorithm as a structurally determined action informed by a lived history of structural coupling with traditional school mathematics. To Lyle, this execution of an algorithm and the answer it yielded was deemed "good enough" to "survive" in school. It constituted what Lyle needed to do to achieve harmony with his mathematics learning environment.

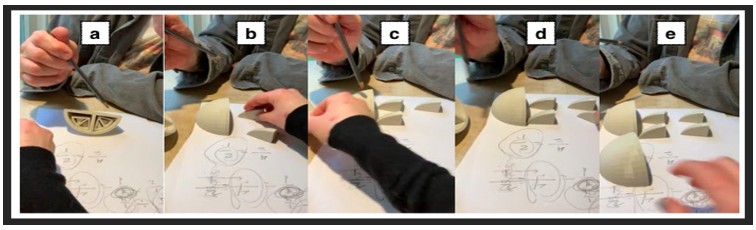
Next, Dolly directed Lyle's attention to the Orange and asked, "Can you show me with this?" From there, the pair set off to navigate a complex interplay of signs literally at hand. Our semiotic analysis enabled us to identify moments of semiotic interference (Maffia & Maracci, 2019) that they experienced as they pursued a non-linear path of problem-solving activity punctuated by moments of what we referred to as either *harmony*, a pleasing fit, or *dissonance*, a displeasing conflict or lack of fit. Next we present one of those moments.

3.3.2.2 A crowning achievement

Here we present what appeared to be a crowning achievement for Dolly and Lyle in their search for harmony in meanings for fraction division mediated by the two artifacts. By enchaining signs (Presmeg, 2006; Bartolini Bussi & Mariotti, 2008) across pieces of the Orange and elements of the algorithm – specifically by translating interpretations of parts of the Orange to interpretations of quantities in the algorithm (i.e. ½ and ¼) – they made sense of those quantities. Then, they engaged in similar sense making in order to find interpretations for the ½ and ¼ in the posed problem, ½ ÷ ¼.

> *Dolly:* <*referring to* ½ ÷ ¼> We wanna take a half of one and divide it by a quarter of one, right?
> *Lyle:* Yes.
> *Dolly:* Take a half of one and divide – oh, that's what it is!
> *Lyle:* It's 2.
> *Dolly:* We wanna take this <points to the half piece of the orange> and see how many of those <*now pointing to quarter piece*> fit in there <*points to the half piece again. Then, with confidence:*> And that's why our answer is 2.
> *Lyle:* Yes.
> *Dolly:* There's still two halves in a whole, 'cuz this <*the expression, ½ ÷ ¼*> is in regards to a whole. <*rephrasing*> This is in regards to 1. So a half of 1 divided by a quarter of 1 is 2, because 2 quarters fit into 1 half. Or <*returning to the expression,* > 4 quarters fit into 2 halves.
> *Lyle:* Yeah.

In this excerpt, we observed the meaning Dolly makes of the expression, ½ ÷ ¼, by enchaining interpretations of ½ and ¼ in light of the measurement meaning of division she and Lyle enacted earlier, as well as the meanings they enacted for ½ and ¼ in the algorithm. Next, Lyle re-enacted the interpretation for himself.



Figures 9a - 9e. Lyle re-enacts Dolly's understanding of "4/2 = 2/1."

Lyle: So this is half of a whole and this is a quarter of a whole. <*Next, he turns his attention to the orange (Figure 9a) and points to the half piece resting on the paper.*> Half of a whole. <*Next, he takes his pencil and points to each quarter piece*:> Quarter of a whole <*Then, pointing to the two quarter pieces:*> is 2.

Dolly: <*pointing to the 2 quarter pieces*> Yeah, 'cause there's two quarters of a whole.

Lyle: Yeah, that makes sense.

Dolly: 'Cause there's two of these <*She pulls out the quarter pieces and sets them next to the half piece (Figure 9b).*> for every one of these <*she says as she touches the half piece*>.

Lyle: <*with a sigh, perhaps of relief*> Yes.

Dolly: Or there's four of these. <*She takes the quarter pieces out of the other half piece.*>

Lyle: <*points to the half piece and extends Dolly's thinking (Figure 9c*)>: For two of those.

Dolly: <*revoicing Lyle*> For two of those. <*As she speaks, she aligns all of the quarter pieces as well as the second half piece on the page (Figures 9d and 9e*).>

As if to establish his own meanings for fraction division and its coherence in representations across artifacts as Dolly had just done, Lyle used the pencil to re-enact a physical bridge between the elements of the problem (½ ÷ ¼) and the pieces of the Orange. He uttered "half of a whole" as he pointed to the ½ on paper, and "quarter of a whole" as he pointed to the ¼. Then he repeated these phrases on the other side of the bridge: "half of a whole" as he pointed to the half piece, and "quarter of a whole" as he pointed to the quarter piece. We interpreted this activity as a matching of his interpretation of half of a whole and quarter of a whole in the symbolic representations (½ and ¼, respectively) to the representations he identified in the orange (the half piece and the quarter piece, respectively). These embodied epistemic actions seem to reify the harmony that had finally emerged from recursive interactions that culminated in an enchaining of signs signifying the sense he and Dolly had made. This reification can be viewed as a newly coupled structure of fraction division for Dolly and Lyle, one that offers a stark contrast to the structurally determined response to fraction division that they enacted at the outset of their activity.

3.3.3 Implications

In analyzing the iterative cycles of harmony and dissonance experienced by Dolly and Lyle, enactivist and semiotic analyses enabled us to see the apparent structural coupling they had with traditional school mathematics and that constituted their felt experiences throughout their drive for fit. In light of this finding, we offer recommendations for pedagogical and material resources in mathematics classrooms that enable, support, and honor this sort of loosely structured problem-solving activity. As Proulx (2013) reminds us, students' paths of problem solving emerge in interactions with the environment and are contingent on their particular mathematical structures and interactions. "Average" paths and tools presumed viable for sense making simply cannot be determined *a priori*. Rather, resources should be provided that are responsive to students' creative and agentive efforts at sense making.

4 Conclusion

The Making experience at the center of this body of work had prospective teachers of elementary mathematics innovating at the intersection of mathematics, pedagogy, and design. That experience provided them with an opportunity to consider the interplay between the iterative design of an evolving artifact and the application of teacher knowledge domains in the artifact's development. In addressing the broadest question, *What are the potential benefits of a Making experience within mathematics teacher preparation?*, our research has revealed a number of positive outcomes. These findings and their implications for teacher learning have been shared at the conclusion of each of the vignettes we presented above. We provide only a summary overview of them here.

Over and over, our findings demonstrate the formative value of immersing prospective teachers in a communal design environment of collective social making and tasking them with a pedagogically genuine design experience centered on the Making of an original physical manipulative for mathematics teaching and learning. In particular, these findings show that the experience informed the pedagogical, mathematical, and design thinking of prospective teachers, while also demonstrating that identity formation (e.g., as a mathematics teacher) is just as central to their learning to teach mathematics as those three forms of thinking. That revelation in turn allowed us to determine that the experience supported prospective teachers' movement along a trajectory of participation that we called a "teacher becoming," with the potential cultivation of a caring pedagogy and of knowledge about the formative power of embodied activity in sense making as just two aspects of that becoming. Lastly, we shared evidence of the experience's potential impact beyond teacher preparation in that it yielded epistemic scaffolds in material form that can support the connection between teacher preparation and teachers' practice.

All in all, we propose that these findings contribute to the bodies of research on both teacher learning and identify formation in teacher preparation. They also generate new opportunities for research that moves the field forward regarding the potential value of constructionist, STEAM-integrated curricular experiences in teacher preparation. Future research could more closely explore the design of these environments in teacher preparation, the teacher educator's role in designing and facilitating these experiences, and the subsequent in-service instruction of teachers who participated in these experiences during teacher preparation.

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